DIPLOMARBEIT

Investigation of complex spatial mode structures of Photons

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Juni 2012
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1. Introduction

Photons are excellent carriers of information in quantum communication or quantum computation. They interact with the environment very weakly, can be controlled very precisely and information can be encoded in several different degrees of freedom such as polarisation or frequency. Furthermore, photons can also have complex spatial structures. This additional degree of freedom leads to a number of fascinating properties such as phase vortices and singularities, orbital angular momentum and the possibility to access higher-dimensional Hilbert spaces. Those complex structures and their properties are the scope of this thesis.

After classifying modes with higher-order spatial structure in chapter 2, we analyse one specific family – namely the Ince-Gauss modes – in chapter 3. They possess a complex vortex- and singularity structure and we believe that they could open up exciting possibilities for new fundamental quantum experiments as well as for novel methods in quantum cryptography and quantum information science. We analyse their phase-vortex behaviour and show for the first time the entanglement of Ince-Gauss modes. We also measure an Ince-Gauss specific quantum correlation function, which could be used in enhanced quantum communication protocols. An idea for a very simple protocol that takes advantage of this correlation function can be found in the appendix.

In the 4th chapter, we consider a special type of artificial beams, which we call the Maverick beams. We construct and analyse them for the first time, and observe very fascinating intensity distribution of the generated beams, such as spiral patterns or smooth generation of radial rings. With a special method of creating an artificial Bloch sphere, we are able to perform the first quantum experiments with Maverick beams. We show the entanglement of several members of these beams, and analyse the quantum correlation of many other cases. It turns out that the quantum correlation depends on the symmetry of the phase structure of the mode.
2. Hierarchy of spatial modes

2.1. From Maxwell’s equations to the paraxial wave equation

Electric and magnetic fields are described by Maxwell’s equations in vacuum [1]

\[ \partial_i E_i = \frac{\rho}{\epsilon_0}, \tag{2.1} \]
\[ \partial_i B_i = 0, \tag{2.2} \]
\[ \epsilon_{ijk} \partial_j E_k = -\partial_t B_i, \tag{2.3} \]
\[ \epsilon_{ijk} \partial_j B_k = \mu_0 J_i + \mu_0 \epsilon_0 \partial_t E_i. \tag{2.4} \]

where \( E_i \) and \( B_i \) are the electric and magnetic fields respectively, \( \epsilon_0 \) and \( \mu_0 \) are the vacuum permittivity and permeability, \( \rho \) and \( J_i \) are the electric charge and the current. In the absence of charges and currents (\( \rho = 0 \) and \( J_i = 0 \)), the Maxwell equations simplify to

\[ \partial_i E_i = 0, \tag{2.5} \]
\[ \partial_i B_i = 0, \tag{2.6} \]
\[ \epsilon_{ijk} \partial_j E_k = -\partial_t B_i, \tag{2.7} \]
\[ \epsilon_{ijk} \partial_j B_k = \mu_0 \epsilon_0 \partial_t E_i. \tag{2.8} \]

To get a wave equation, we perform a simple and well-known calculation. We applying the curl operation to (2.7), which leads to

\[ \epsilon_{ghi} \partial_h \epsilon_{ijk} \partial_j E_k = -\epsilon_{ghi} \partial_h \partial_t B_i. \tag{2.9} \]

Using the equation (2.8) and \( \epsilon_{ghi} \epsilon_{ijk} = \delta_{gh} \delta_{jk} - \delta_{gi} \delta_{hk} \) simplify (2.9) to

\[ (\partial_g \partial_h) E_h - (\partial_h \partial_h) E_g = -\partial_t \mu_0 \epsilon_0 \partial_t E_g. \tag{2.10} \]

Because of equation (2.5), we can further simplify to

\[ -(\partial_h \partial_h) E_g = -\frac{1}{c^2} \partial_t^2 E_g, \tag{2.11} \]

where \( \frac{1}{c^2} = \mu_0 \epsilon_0 \), and \( c \) is the vacuum speed of light. Using the D’Alembert operator \( \Box := \frac{1}{c^2} \partial_t^2 - \partial_i^2 \), we can write the wave equation for electromagnetic waves in a very compact form:

\[ \Box E(\mathbf{r}, t) = 0. \tag{2.12} \]

A similar method can be applied to the magnetic field.
2.2 Classification of spatial modes

For calculating the paraxial wave equation, we follow [2]. The field $E(r,t)$ should satisfy the equation (2.12). We can separate the spatial and temporal coordinates of $E(r,t) = E(r) \exp(i\omega t)$, which leads to

$$(\Delta + k^2)E(r,t) = 0,$$  \hspace{1cm} (2.13)

where $k = \frac{\omega}{c}$ is called the wave number. This equation (2.13) is called Helmholtz equation. We can further simplify the E-field by restricting to cases where

$$E(r) = A(r) \exp(ikz).$$  \hspace{1cm} (2.14)

This is called the paraxial approximation. The electric field is written as a product of a complex amplitude $A(r)$ and a sinusoidal plane wave in the longitudinal component. This approximation requires that $A(r)$ varies just slowly in $z$-direction, which can be expressed as

$$\left| \frac{\partial A}{\partial z} \right| \ll \left| kA \right|,$$  \hspace{1cm} (2.15)

$$\left| \frac{\partial^2 A}{\partial z^2} \right| \ll \left| k^2 A \right|.$$  \hspace{1cm} (2.16)

By a substitution of (2.14) to (2.13), we get

$$\left( \Delta_\perp + 2ik \frac{\partial}{\partial z} \right) A(r) = 0$$  \hspace{1cm} (2.17)

where $\Delta_\perp$ is the Laplacian in the transverse plane. This equation is called the Paraxial Wave Equation (PWE), and is of great importance in optics, as it describes many practical situations such as the radiation of a laser.

2.2. Classification of spatial modes

There are many different solutions to the PWE (2.17), which lead to many different spatial modes of light. A classification of those different solutions is required to understand the connections between different families of modes and their possible capability for fundamental experiments or applications.

One possibility to classify the solutions of the equation (2.17) is given here.

1. We separate the solutions using the degree of the orthogonal coordinate system, from which they emerge [3]. The degree of the coordinate system refers to the degree of the polynomial that describes the system. We restrict ourselves to propagation-invariant optical fields (PIOFs - fields that do not change their transverse intensity distribution while propagating), which fulfill the PWE. Therefor the $z$-axis of the coordinate system has to be cartesian. This simplifies our task to find coordinate systems in the 2-dimensional transverse-plane. An example of a coordinate system described by a polynomial of degree one is the Cartesian system; polar, elliptical and parabolic coordinate systems are described by a polynomial of degree two. A coordinate system of degree three would be defined by elliptical functions (not elliptical coordinates, which are of degree two).
2.2 Classification of spatial modes

2. We divide the solutions into those which are derived by using full separation of all coordinates and those which are not derived by separation of variable ansatz. The separation ansatz of the PWE (which is a partial differential equation (PDE) in three coordinates) leads to three ordinary differential equations (ODEs). It is easier to solve ODEs than PDE, therefore its solutions are much explored. Examples are the well-known TEM laser-modes (Hermite-Gauss modes), which emerge as a natural solution from a separated cartesian coordinate system. An example for a non-separated solution of the PWE are Airy-beams, which are derived using the ansatz $U(r) = U_x(x, z)U_y(y, z)$. Those beams change their shape in the longitudinal direction [4][5].

3. The next division concerns the coordinate system which is used to derive the solutions: There are two general orthogonal coordinate systems of degree two that can describe a 2-dimensional plane: the parabolic and the elliptical coordinate system. Beams derived from separated parabolic coordinates are called Parabolic Beams [6]. Very general solutions in elliptical coordinates have been derived in 2008 [7], which contain the special cases of cartesian [8] and circular beams [9]. One very general family of solutions in elliptical coordinates are Ince-Gauss modes [10][11], which covers several other families of solutions as a special case.

4. In this thesis, we are interested especially in Ince-Gauss modes and their special cases. One special case are Mathieu-Gauss beams [12], which were the first discovered family of elliptical modes. They have an infinite number of rings with a Gaussian envelope. Other special cases of the Ince-Gauss modes are the Laguerre-Gauss modes [13], which appear when the elliptic coordinate system becomes a polar coordinate system for vanishing ellipticity. At the limit of infinite ellipticity, the other special emerges, which are the natural solutions in cartesian coordinates called Hermite-Gauss modes.

Figure (2.1) shows a compact form of the classification. It should be noted that there are different ways to classify those modes. However, taking advantage of the natural solutions in different coordinate systems simultaneously classifies modes with different physical properties. For example, we separate between modes from elliptical and parabolical coordinate systems. The modes from elliptical coordinate system have integer characterising numbers; modes with different integer numbers are orthogonal. However, parabolic beams have a continuous characterising parameter, and modes with a different parameter are orthogonal.
2.2 Classification of spatial modes

Figure 2.1: Classification of solutions to the paraxial wave equation.

2.2.1. Laguerre-Gauss modes

The Laguerre-Gauss family is an important set among spatial modes [14]. These beams can be written (in the $z=0$ plane, where $z$ is the propagation direction) as

$$LG_{p,l}(r,\phi) = \sqrt{\frac{2p!}{\pi(p+|l|)!}} \cdot \frac{1}{w_0} \cdot \left(\frac{\sqrt{2r}}{w_0}\right)^{|l|} \cdot \exp\left(-\frac{r^2}{w_0^2}\right) \cdot L_p^{(|l|)}\left(\frac{2r^2}{w_0^2}\right) \cdot \exp(il\phi), \quad (2.18)$$

where $p \in \mathbb{N}$ and $l \in \mathbb{Z}$ are the characterizing integer numbers of the mode. The $l$ corresponds to the phase-vortices of the mode, for $l > 0$, they have a phase singularity and an ring-shaped intensity with an intensity zero in the center. The $p$ corresponds to additional rings. The $w_0$ is the beam waist at $z=0$, and $L_p^{(|l|)}$ are the generalized Laguerre polynomials. A set of intensity and phase structure of different Laguerre-Gauss modes is shown in Figure 2.2.

A phase singularity is a point of undefined phase. The phase circulates around these points of zero intensity - therefore they are called Vortices.
2.3 Artificial beams

The classification in chapter (2.2) does not comprise all possible modes. Some families of modes have been created artificially, in contrast to direct derivation from the PWE. They have been used to discover general properties of vortices or phase-structures.

2.3.1. Fractionalization of optical beams

Fractionalization is a continuation of the characterizing integer parameters to real numbers. These methods are used to better understand the fundamental meaning of their quantum numbers and the connection to physical quantities.
2.3 Artificial beams

Trivial fractional OAM beams

A simple way for continuation of a integer quantum number is to use (2.18) with p=0, which corresponds to one single ring. These simple fractional OAM beams are defined with $l \in \mathbb{R}$. Their vortex dynamics has been studied from different aspects. It has been shown theoretically [25] and experimentally [26], that for specific values of $l$ a singular vortex-line appears. Surprisingly, the OAM content of those beams is not linear in $l$, but depends on a more complicated non-linear function.

![Figure 2.3: Left: Phase pattern of a $LG_{0,2}$ beam. Two $2\pi$ phase jump can be seen. One extra unnatural phase jump appears in horizontal direction from the center to the right side. Right: Density plot for $|LG_{0,2,2}|$ in the z=0 plane. In addition to the central intensity zero, a horizontal zero-intensity line can be observed at the position of the extra phase jump. (Source for the right picture: [25])](image)

In a different experiment a diffraction methods has been used, which extracts the degree of a vortex in standard LG modes [27]. They used a triangular apperture to analyse the diffraction pattern of a simple fractional OAM beam, and visualized the birth of an optical vortex [28].

Fractional continuation

Developing fractional order modes can be achieved in an analytical way too, as it is shown in [29][30]. There, Gutiérrez-Vega develops a fascinating method by using $(\hat{L}u = 0$ and $[\hat{D}, \hat{L}] = 0) \Rightarrow \hat{L}(\hat{D}u) = 0$, and the property that a higher order solution can be created from a lower order solution using a derivatives with respect to a certain coordinate system.

Instead of integer order derivatives, the author uses the Riemann-Liouville $\alpha$-th-order fractional differential operator, which leads to "natural" fractional mode with real numbers of $l$ and $p$, which are OAM and number of additional rings respectively.
2.3 Artificial beams

Figure 2.4: For different integer values of \( l \), the continuous transition of the radial index \( \eta \) can be observed. The continuous development of additional radial rings can be observed. (Note that the index \( \eta \) does not directly correspond to the number of rings in LG modes, as the paper considers elegant Laguerre-Gauss modes. Source: [30])

The continuous transitions shows in an analytical way how new vortices and new rings appear, and how the symmetry of the phase- and intensity-pattern changes.

2.3.2. Maverick beams

Another example of artificial modes are the Maverick beams (MAVs), which we have generated and demonstrated entanglement. They are described in detail in chapter (4). These modes are can be seen as an artificial generalisation of Laguerre-Gauss modes. The phase structure of LG modes always has \( p \) radial phase-jumps of \( \pi \), and a linear phase-change from \([0, l \cdot 2\pi]\). The MAVs are constructed from the phase-patterns only. In contrast to LG modes, they can have different \( l \) in every ring, they can have a different phase-jump at each boarder of a ring (not only a phase-jump of \( \pi \)), and their angular phase-change can be non-linear. These generalisations lead to a very big set of possible modes, including many new features which show interesting characteristics and properties of general phase-patterns and their resulting modes.
3. Ince-Gauss modes

3.1. Derivation of Ince-Gauss modes

3.1.1. Elliptic coordinate system

Ince-Gauss modes are solutions of the PWE (2.17) in elliptical coordinates. The 2-dimensional elliptic coordinate system is described by the radial elliptic coordinate $u$, and the angular elliptic coordinate $v$. The transformation between elliptical $(u, v)$ and cartesian $(x, y)$ coordinates is given by

$$
\begin{pmatrix}
x \\
y
\end{pmatrix}
= f_0 \cdot \begin{pmatrix}
\cosh u \cdot \cos v \\
\sinh u \cdot \sin v
\end{pmatrix}
$$

(3.1)

$f_0$ is the semi-focal separation (eccentricity) of the coordinate system. The coordinate lines are confocal ellipses and hyperbolae. In the limit of $f_0 \to 0$, the well-known polar coordinate system emerge. Figure (3.1) shows an elliptical coordinate grid.

Figure 3.1: Elliptical coordinate system grid. The confocal ellipses and hyperbolae (green) represent the coordinate lines. The common foci define the eccentricity. The black ellipse emerges for $u = 1$ and $v \in [0, 2\pi)$. (Source: Wikimedia, Creative Commons license)
3.1 Derivation of Ince-Gauss modes

3.1.2. Solution of the Ince-Equations

The Ince-Gauss modes can be derived by separation of variables in an elliptical coordinate system (3.1) of the paraxial wave equation (2.17). Here we follow the derivation by Arscott [31].

For simplicity we perform the calculations in the waist plane \( z=0 \). A separation ansatz is used to solve the PWE in elliptical coordinates

\[
A(\mathbf{r}, z) = E(u) \cdot N(v) \cdot \Psi_G(\mathbf{r})
\]  

where \( \Psi_G(\mathbf{r}) = \exp(-\frac{r^2}{w_0^2}) \) is a Gaussian envelope. This leads to the following ordinary differential equations

\[
\frac{d^2 E}{du^2} - \epsilon \cdot \sinh(2u) \cdot \frac{dE}{du} - \left( a - p \cdot \epsilon \cdot \cosh(2u) \right) \cdot E = 0,
\]  

\[
\frac{d^2 N}{dv^2} + \epsilon \cdot \sin(2v) \cdot \frac{dN}{dv} - \left( a - p \cdot \epsilon \cdot \cos(2v) \right) \cdot N = 0,
\]

where \( \epsilon = \frac{2f^2}{w_0^2} \) is the ellipticity parameter, \( w_0 \) is the beam size at the waist and \( a \) and \( p \) are separation constants. Equation \( (3.3) \) can be transformed to \( (3.4) \) by \( u = iv \), which makes \( (3.4) \) the only ordinary differential equation, which needs to be solved. This equation is called the Ince-Equation, and has been studied for the first time by the Edward Lindsay Ince in the 1920s [32].

The Ince-Equation can be solved using one of four general infinite-sum ansatze

\[
N(v) := C_e(v) = \sum_{r=0}^{\infty} A_r \cos(2r \cdot v),
\]

\[
N(v) := C_o(v) = \sum_{r=0}^{\infty} A_r \cos((2r+1) \cdot v),
\]

\[
N(v) := S_e(v) = \sum_{r=1}^{\infty} A_r \sin(2r \cdot v),
\]

\[
N(v) := S_o(v) = \sum_{r=0}^{\infty} A_r \sin((2r+1) \cdot v).
\]

This leads to the Ince-polynomials, where \( C(v) \) and \( S(v) \) stands for even and odd Ince-polynomials, and the subscript denotes an even or odd argument. To solve the Ince-Equation, we use the first ansatz \( (3.5) \); the solution with another ansatz is very similar.

We compute the zeroth, first and second derivative of \( N(v) \)

\[
N(v) = \sum_{r=0}^{\infty} A_r \cos(2r \cdot v)
\]

\[
\frac{dN}{dv} = -\sum_{r=0}^{\infty} A_r (2r) \sin(2r \cdot v)
\]

\[
\frac{d^2 N}{dv^2} = -\sum_{r=0}^{\infty} A_r (2r)^2 \cos(2r \cdot v)
\]
By using the following properties of trigonometric functions

\[
\cos(2 \cdot z) \cdot \cos(2r \cdot z) = \frac{1}{2} \left( \cos (2 \cdot (r-1) \cdot z) + \cos (2 \cdot (r+1) \cdot z) \right) \tag{3.12}
\]

\[
\sin(2 \cdot z) \cdot \sin(2r \cdot z) = \frac{1}{2} \left( \cos (2 \cdot (r-1) \cdot z) - \cos (2 \cdot (r+1) \cdot z) \right) \tag{3.13}
\]

one can rewrite the three terms of (3.4)

\[
T_1 = - \sum_{r=0}^{\infty} A_r \cdot (4 \cdot r^2) \cdot \cos (2 \cdot r \cdot z) \tag{3.14}
\]

\[
T_2 = - \sum_{r=0}^{\infty} \epsilon \cdot r \cdot A_r \cdot \left( \cos (2(r-1) \cdot z) - \cos (2(r+1) \cdot z) \right) \tag{3.15}
\]

\[
T_3 = \sum_{r=0}^{\infty} A_r \cdot \left( a \cdot \cos (2r \cdot z - \frac{\epsilon}{2} \cdot p \cdot \left( \cos (2(r-1) \cdot z) + \cos (2(r+1) \cdot z) \right)) \right) \tag{3.16}
\]

Inserting the three terms into (3.4) and reordering them gives

\[
\sum_{r=0}^{\infty} \left( - \frac{\epsilon}{2} \cdot (p + 2r) \cdot A_r \cdot \cos (2(r-1)z) + (4r^2 - a) \cdot A_r \cdot \cos (2r \cdot z) + \frac{\epsilon}{2} \cdot (p - 2r) \cdot A_r \cdot \cos (2(r+1)z) \right) = 0 \tag{3.17}
\]

As \( \cos(nx) \) form an orthogonal basis, the equation has to be fulfilled for each \( r \) – which defines the following recurrence relation for the coefficients \( A_r \):

\[
\cos (2r \cdot z) = \cos (nz)
\]

\[
n = 0 : \quad \frac{\epsilon}{2} \cdot (p + 2) \cdot A_1 + (0 - a) \cdot A_0 = 0
\]

\[
n = 2 : \quad \frac{\epsilon}{2} \cdot (p + 4) \cdot A_2 + (4 - a) \cdot A_1 + \frac{\epsilon}{2} \cdot (2p - 0) \cdot A_0 = 0
\]

\[
n = 4 : \quad \frac{\epsilon}{2} \cdot (p + 6) \cdot A_4 + (16 - a) \cdot A_3 + \frac{\epsilon}{2} \cdot (p - 0) \cdot A_1 = 0
\]

\[
n = 6 : \quad \frac{\epsilon}{2} \cdot (p + 8) \cdot A_6 + (36 - a) \cdot A_4 + \frac{\epsilon}{2} \cdot (p - 4) \cdot A_2 = 0
\]

\[
\ldots
\]

For \( p = 2n \), finite even solutions can be obtained

\[
(0 - a) \cdot A_0 + \epsilon \cdot (n + 1) \cdot A_1 = 0
\]

\[
2 \epsilon \cdot n \cdot A_0 + (4 - a) \cdot A_1 + \epsilon \cdot (n + 2) \cdot A_2 = 0
\]

\[
\epsilon \cdot (n - r) \cdot A_r + (4(r+1)^2 - a) \cdot A_{r+1} + \epsilon \cdot (n + r + 2) \cdot A_{r+2} = 0, \quad r = 1, \ldots, n - 2
\]

\[
\epsilon \cdot A_{n-1} + (4n^2 - a) \cdot A_n = 0
\]

The recurrence relation can be rewritten as an eigenvalue problem

\[
M \cdot A_i = a_i \cdot A_i \tag{3.19}
\]
3.2 Classical properties

\[
M = \begin{pmatrix}
0 & \epsilon \cdot (n+1) & 0 & 0 & 0 & \cdots & 0 \\
2\epsilon \cdot n & 4 & \epsilon \cdot (n+2) & 0 & 0 & \cdots & 0 \\
0 & \epsilon \cdot (n-1) & 16 & \epsilon \cdot (n+3) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 2\epsilon & 4(n-1)^2 & 2n \cdot \epsilon \\
0 & 0 & \cdots & 0 & 0 & \epsilon & 4n^2 \\
\end{pmatrix} \quad (3.20)
\]

The eigenvalues of \( M \) correspond to the \( a \) parameter in the differential equation, and the eigenvectors are the coefficients \( A_i \) for the ansatz.

The Ince-polynomials are defined via two parameters:

1. The parameter \( p \), which has to be an integer number due to the recurrence relation in (3.18).
2. The parameter \( m \in \mathbb{N} \), which stands for the \( m \)-th eigenvalue \( a_m \) of the eigensystem (3.19).

Both parameters have either be even or odd, and \( p \leq m \).

Inserting the Ince-polynomials into (3.2) leads to the even and odd Ince-Gauss modes

\[
IG^e_{p,m,\epsilon}(r) = N_e \cdot C^m_p(i \cdot u, \epsilon) \cdot C^m_p(v, \epsilon) \cdot e^{-\frac{r^2}{w_0^2}},
\]

\[
IG^o_{p,m,\epsilon}(r) = N_o \cdot S^m_p(i \cdot u, \epsilon) \cdot S^m_p(v, \epsilon) \cdot e^{-\frac{r^2}{w_0^2}}.
\]

(3.21)

(3.22)

For equal ellipticity, modes with a different number of \( p \) or \( m \) are orthogonal. As they can take every value in \([0, \infty)\), they define an infinite dimensional Hilbert space. \( N_e \) and \( N_o \) are normalisation constants. For the special case for \( \epsilon \to \infty \), the helical Ince-Gauss modes can be defined as superposition of even and odd Ince-Gauss modes [33]

\[
IG^\pm_{p,m,\epsilon}(r) = \frac{1}{\sqrt{2}} \left( IG^e_{p,m,\epsilon}(r) \pm i \cdot IG^o_{p,m,\epsilon}(r) \right).
\]

(3.23)

In this thesis, we will mainly consider helical Ince-Gauss modes, therefor further refere them just as Ince-Gauss modes.

3.2. Classical properties

For every constant eccentricity \( f_0 \), the elliptical coordinate system is a complete basis for the 2-dimensional space. Similarly, for a constant ellipticity \( \epsilon \) the Ince-Gauss modes form a complete orthonormal basis. The phase and intensity distribution of Ince-Gauss modes with a constant ellipticity \( \epsilon = 2 \) can be seen in Figure (3.2).
3.2 Classical properties

Figure 3.2: Six different members of Ince-Gauss modes with constant ellipticity $\epsilon = 2$. The upper row shows the intensity distribution, while the lower row shows the phase pattern. From left to right: IG$_{2,2,2}$, IG$_{3,3,2}$, IG$_{3,1,2}$, IG$_{4,2,2}$, IG$_{5,3,2}$, IG$_{8,4,2}$. For each value of $\epsilon \in \mathbb{R}^+$, the Ince-Gauss modes define a different complete orthonormal basis, therefore the shown modes are six out of infinite orthogonal basis functions.

In the limit $\epsilon \to 0$, Ince-Gauss modes become Laguerre-Gauss modes with an integer OAM value, and all vortices moving to the centre of the beam. In the limit of $\epsilon \to \infty$, the Ince-Gauss modes become "helical" Hermite-Gauss modes [33], superpositions of two HG modes. This transition can be seen in Figure (3.3).

Figure 3.3: Ince-Gauss modes IG$_{5,3,\epsilon}$ with two rings and three vortices for different values of $\epsilon$. In the upper row the intensity distribution of each mode is plotted, in the lower row the phase distribution is presented. From left to right the values of $\epsilon$ are (0; 0.5; 1.0; 2.0; 5.0; $\infty$). The splitting of the vortex in the centre into three vortices on a horizontal line can be observed as well as the creation of additional pairs of vortices in the ring of zero intensity [34]. For $\epsilon = 0$, the IG beam becomes a Laguerre-Gauss mode with continuous rotational symmetry of the intensity pattern. For $\epsilon > 0$, only a 2-fold rotational symmetry remains. In the limit of $\epsilon \to \infty$, a 4-fold rotational symmetry emerges, with the corresponding modes being called helical Hermit-Gauss modes [35].
3.2 Classical properties

The link to the well-known LG modes becomes clear when the two characterizing parameters are studied in more detail. The number of vortices, and thereby the corresponding topological charge \( l \) for the Laguerre-Gauss case, is given by the parameter \( m \in [0, \infty) \) and is called the order of the mode. The other parameter \( p \in [0, \infty) \) is called degree of the mode. Together with \( m \) it defines the number of additional rings \( n \in \mathbb{N} \) in Laguerre-Gauss modes by the expression

\[
n = \frac{p^2 - m^2}{2}.
\]

### 3.2.1. OAM of Ince-Gauss modes

In 2006 it has been shown that particles which interact with an elliptical Mathieu-Gauss beam (a special case of Ince-Gauss beams) start to rotate, their trajectories are ellipses [36]. The orbital angular momentum of those beams has been calculated using a generalisation of the orbital angular momentum operator [37]. The authors define the average OAM per photon \( L_z \) as an expectation value

\[
\delta L_z = \frac{\int \int \delta l_z(x,y) \cdot |u(x,y)|^2 dxdy}{\int \int |u(x,y)|^2 dxdy} \quad (3.24)
\]

of a local OAM per photon

\[
\delta l_z = \frac{d}{d\phi} \arg(u(x,y)) \cdot \hbar, \quad (3.25)
\]

where \( u(x,y) \) denotes the amplitude of the mode in the transverse plane. For Laguerre-Gauss modes (2.18), this formulas give

\[
\delta l_z = \frac{d}{d\phi} \arg(LG(x,y)) \cdot \hbar \quad (3.26)
\]

\[
= \frac{d}{d\phi} \arg\left(\exp(i \cdot l \cdot \phi)\right) \cdot \hbar \quad (3.27)
\]

\[
= \frac{d}{d\phi} (l \cdot \phi) \cdot \hbar = l \cdot \hbar \Rightarrow \quad (3.28)
\]

\[
\delta L_z = \frac{\int \int l \cdot \hbar \cdot |u(x,y)|^2 dxdy}{\int \int |u(x,y)|^2 dxdy} = l \cdot \hbar \quad (3.29)
\]

The same procedure can be applied to Ince-Gauss modes to analyse the behaviour of the average OAM per photon when the ellipticity increases. This has been done in [34], and as a first step we want to reproduce that result. Figure (3.4) shows the numerically calculated OAM per photon \( L_z \) for different values of the ellipticity \( \epsilon \).
Figure 3.4: Average OAM per photon (3.24) for different values of ellipticity. In the left plot, the IG modes do not have additional rings, they just possess a different number of vortices. It can be observed that the average OAM decreases when $\epsilon$ increases. The right picture shows the behaviour of modes with more than one ring. The surprising effect is that the OAM decreases for small ellipticities, but increases again at some critical value. A similar effect has been observed in Mathieu-beams where the authors connect the increasing of the OAM to the generation of additional vortices in the outer parts of the mode [37].

The formula (3.24) contains the intensity as well as the angular change of the phase of the mode, therefore analysing these two values together can give new insights. In figure (3.5) we combine these values, by encoding the intensity into color-information, and the angular derivative of the phase is given by the 3D-structur of the plot.
3.2 Classical properties

Figure 3.5: The intensity and change of the phase plotted together for several modes. The intensity is encoded in color-information, the local OAM per photon $\delta l_z$ (3.25) is encoded as the shape of the plot. The x- and y-axis correspond to the transverse plane, the z-axis is the angular derivative of the phase (in arbitrary units). It can be seen that the central phase-singularity splits when the ellipticity increases. New vortices appear in the outer intensity-zero rings. Singularities that correspond to delta-functions (as for $\epsilon = 0$) are not shown because of numerical issues in the computation.

It can be noticed that the intensity would flow to regions where the phase-change is small.

Further investigations in the classical behaviour of the vortices and its connection to physical properties are worthwhile.

3.2.2. Triangular-aperture method

There have been proposals and experiments for measuring the OAM of Laguerre-Gauss modes, by using conformal phase transformation of the mode [38], rotational Doppler shift [39], mode specific detection [16], using a Dove prism [40], performing off-axis measurements of the mode [41], and several interference methods [42][43]. We will focus on the method described in [27]. The authors discovered that a Laguerre-Gauss going through a triangular aperture creates a triangle diffraction pattern. The triangle diffraction pattern consists of $l+1$ dots per side, and the orientation of the pattern exposes the sign of $l$. Figure (3.6) shows their main result.
3.2 Classical properties

Figure 3.6: The triangular-method to retrieve the OAM of a mode. \( m \) stands for the topological charge/OAM of the mode. (Source: [27]).

Ince-Gauss modes have split vortices. One could ask what is the topological charge of one single vortex, and it turns out that this triangular-method gives an interesting experimental clue about the answer.

We built a simple triangular diffraction apperture made from razor blades. First we verified that the method works for Laguerre-Gauss modes (Figure 3.7).

Then we apply the same method to Ince-Gauss modes by holding it over one single vortex of a \( \text{IG}_{2,2,5} \) mode, as it can be seen in Figure 3.8.

Figure 3.7: Verification of the triangular-diffraction method using LG modes. The diffraction patterns shows the characteristic triangular shape with \( l+1 \) dots (in analogy to the original paper, we denote the topological charge by \( m \)).
3.3 Experimental setup and challenges

In chapter 2.2.1, we have talked about quantum mechanical experiments concerning Laguerre-Gauss modes. In this chapter we show the experimental setup and the measurement results for the entanglement of Ince-Gauss modes. The content of this chapter has been combined to a scientific paper [53].

In our experimental setup (Figure 3.9), we employ a type-II spontaneous parametric down-conversion (SPDC) in a nonlinear crystal (periodically poled potassium titanyl phosphate, ppKTP) which creates pairs of photons, including higher order spatial modes. The two photons are collinear and have orthogonal polarisations. We split the signal and idler photon on a polarizing beam splitter, and analyse them in the two arms of the setup by using a combination of Spatial Light Modulators (SLMs) and single mode fibres (SMFs). An SLM is a liquid crystal display, which can perform an arbitrary phase transformation on the incoming beam depending on the hologram displayed on it. In our experiment we use computer-generated holograms to convert specific higher order modes into a Gauss mode, which we couple into a SMF. Since the SMFs effectively only allow coupling for Gauss modes, we realise in this way a spatial-mode-specific filter. The measurement projects the state into a specific superposition. The photons are then detected with single-photon detectors and pairs are counted via an FPGA-based coincidence-logic. To account for SLM conversion deficiency, we use a diffraction grating, to spatially separate the zeroth order (which is unmodulated) and the first order (which by construction has correct phase modulation). Thus we collect photons of the first order only.
3.3 Experimental setup and challenges

Figure 3.9: Schematic sketch of the experimental setup. We pump a nonlinear ppKTP crystal with a 405nm diode laser, and get 810nm down-converted spatially entangled photons of orthogonal polarisation. We separate the two photons on a polarizing beam splitter (PBS), and manipulate their spatial mode at Spatial Light Modulators (SLMs), which transform specific Ince-Gauss modes into Gauss modes. The half-wave plate (HWP) is used to transform vertical to horizontal polarisation, which is required by the SLM. The photons in the Gauss modes are then filtered by coupling into single mode fibres (SMFs). Finally, they are detected with two avalanche photo-diodes (D) and analysed with a coincidence-logic (&).

The setup in general is simple, still it turns out that there are some details that deserve more attention. In particular, there were three effects that need to be considered before the begin of the actual experiments; they will be explained briefly.

3.3.1. Different correlation surface for ±l holograms

For aligning the hologram on the SLM to the downconverted photons, we take advantage of OAM conservation in the downconversion process. The conservation allows us to write down the downconversion state in terms of Laguerre-Gauss (LG) modes

\[
|\Psi\rangle = \sum_{l=-\infty}^{\infty} a_l |l, -l\rangle
\]

where \(|l, -l\rangle\) stands for a two-photon state with OAM \(l\) and \(-l\) on the two sides. Therefore entangled photons always have the opposite OAM, and no other combinations should be detected in a coincidence-measurement. The mode is written always with respect to the central axis of the coordinate system. A mode written with respect to an off-axis point in
the coordinate system is a decomposition into other modes. For instance the offaxis mode of $|1, -1>$

$$|1, -1 >_{\text{off-axis}} = \sum_{l=-\infty}^{\infty} b_l |l, -l>.$$  (3.31)

For the correct position, all coefficients beside of $b_1$ vanish.

In the experiment we apply this technique by looking at coincidences with a LG$_{0,1}$ hologram on SLM1, and a Gauss hologram LG$_{0,0}$ SLM2. The Gauss hologram is plain, therefore does not depend on a specific position. We minimize the coincidences by changing the x- and y-position of the LG$_{0,1}$ hologram on SLM1. When we reach a minimum, we found the right position. This method works because an off-axis LG$_{0,1}$ beam can be decomposed into other LG basis modes [41], especially there is a non-zero LG$_{0,0}$ term, which is responsible for the measured coincidences.

This technique should work with every other LG mode in the same way. When we started to work with the setup, such anti-correlations have been measured for LG modes with $l=+1$ and $l=-1$. Surprisingly, it turned out that the method gives a different result for the two modes as shown in Figure 3.10.

![Figure 3.10: The coincidences for LG$_{l=1}$ and LG$_{l=-1}$ holograms on SLM1 and Gauss holograms on SLM2. The x- and y-direction represents a position of the LG hologram. Black corresponds to maximum coincidences, while white corresponds to no coincidences. This shows that the method gives a different result for the two modes.](image)

The effect gets bigger when the coupling into the single mode fibres gets worse. One can presume that a similar effect to off-axis decomposition is responsible; this time for coupling into the fiber.

While analysing this problem, it became clear that instead of using a LG hologram for aligning the positions, using the simplest Hermite-Gauss hologram is even better. The simple Hermite-Gauss hologram consists of one single $\pi$-phase-jump in the center, either in horizontal or vertical direction. As Hermite-Gauss modes are a superposition of two Laguerre-Gauss modes, this technique works as well. The advantage is that it fully decouples the x- and y-directions; therefore one can first find a minimum in x-direction, and later in y-direction, which increases the precision.
3.3 Experimental setup and challenges

3.3.2. Ince-Gauss specific singles-counts oscillation

An oscillation of the single photon counts of Ince-Gauss modes has been detected, when scanning through the equator of the Bloch sphere (Figure 3.11).

Figure 3.11: Singles in both arms detected in a measurement of IG\(_{2,2,2}\). In this measurement SLM1 scanned from \(\phi = 0\) to \(\phi = 2\pi\) four times, while the phase \(\phi\) at SLM2 has set to \(\phi = [22.5^\circ, 67.5^\circ, 112.5^\circ, 157.5^\circ]\). It can be seen that the single counts oscillation depends on the phase of the superposition.

This Ince-Gauss specific effect originates directly in the down-conversion: Laguerre-Gauss modes with OAM=±l have the same probability to be created in down-conversion, but each mode with different indices l and p has a different probability. For higher values of p and l, the probability that the mode is created in the down-conversion process decreases [49].

Ince-Gauss modes can be decomposed into Laguerre-Gauss modes. A simple example with IG\(_{2,2,\epsilon}\) is shown here. The helical Ince-Gauss mode is a superposition of even and odd Ince-Gauss modes

\[
IG_{2,2,\epsilon}^\pm = \frac{1}{\sqrt{2}} \left( IG_{2,2,\epsilon}^e \pm IG_{2,2,\epsilon}^o \right). \tag{3.32}
\]

In the decomposition, the even Ince-Gauss mode has one additional term in contrast to the odd mode [11]:

\[
IG_{2,2,\epsilon}^\pm = (a \cdot LG_{0,2}^e + b \cdot LG_{1,0}^e \pm c \cdot LG_{0,2}^o). \tag{3.33}
\]

where \(LG^e/o\) are even and odd Laguerre-Gauss modes. Similar to even and odd Ince-Gauss modes, they can be written as a superposition of (helical) Laguerre-Gauss modes \(LG^\pm\).

When we look at modes at the equator of the Bloch sphere, we write them as a superposition

\[
IG_{2,2,\epsilon}^{\text{equator}}(\phi) = aLG_{0,2}^e + bLG_{1,0}^e + cLG_{0,2}^o + e^{i\phi} \left( aLG_{0,2}^e + bLG_{1,0}^e - cLG_{0,2}^o \right). \tag{3.34}
\]

The modes for \(\phi = 0^\circ\) and \(\phi = 180^\circ\) can be written as

\[
IG_{2,2,\epsilon}^{\text{equator}}(0^\circ) = 2 \cdot c \cdot LG_{0,2}^o; \tag{3.35}
\]

\[
IG_{2,2,\epsilon}^{\text{equator}}(180^\circ) = 2 \left( a \cdot LG_{0,2}^e + b \cdot LG_{1,0}^e \right). \tag{3.36}
\]
The probabilities for $LG_{0,2}^e$ and $LG_{0,2}^o$ are equal due to OAM conservation. If the probability of $LG_{0,2}^{e/o}$ is different to the probability of $LG_{1,0}^e$, then the photon count rate between two different points on the Bloch sphere’s equator are different. This is exactly what we measure.

This effect could be minimized by mode-specific Procrustean filtering [44]. However, it has been measured that this filtering does not change the coincidence counts in a perceptible way, therefore no filtering is applied in the experiment.

### 3.3.3. Systematical deviation from $\sin^2$-fit

When performing a measurement of coincidences at the equator of the Bloch sphere, and fitting the results with the expected $\sin^2$-function, one recognizes residuals which indicates a systematical effect (Figure 3.12).

**Figure 3.12:** Upper row: Coincidence counts for $IG_{2,2,2}$. The superposition hologram at SLM1 has a phase $\phi = [0, 360^\circ]$ in $5^\circ$ steps. SLM2 changes the phase from left to right: $\phi = [22.5^\circ, 67.5^\circ, 112.5^\circ, 157.5^\circ]$. The red dots shows the measured data, the blue line shows a $\sin^2$-fit. The residuals of the fit can be seen in the lower row.

It can be shown that the same effect exists for Laguerre-Gauss modes, therefore it is not specific to the Ince-Gauss modes. This effect leads to unphysical values of the left-hand side of the CHSH inequality. One possible explanation could be effects from higher order modes in the generated infinite-dimensional Hilbert space; a recent research deals with a very similar problem [46].

The problem most likely originates from the plane-wave holograms that we use. They perform the correct transformation for incoming plane-waves. However, our incoming beam is a Gaussian beam, therefore the hologram is an approximation. Using Gaussian mode holograms would be the correct method; this technique has been described in detail in [45].

We have successfully applied Gauss-mode holograms to Laguerre-Gauss modes, which has led to vanishing of systematical deviations of the $\sin^2$ fit and a physically reasonable violation of CHSH inequality.

However, the same method has not yet been implemented for Ince-Gauss modes because of the complexity of the calculations. Nevertheless it is a very reasonable assumption, due to the successful implementation of Gaussian mode holograms for Laguerre-Gauss, that
Gaussian mode holograms will work for Ince-Gauss modes as well. For the latter results, this small systematic effect has been neglected.

### 3.4. Quantum mechanical properties

#### 3.4.1. 2-dimensional entanglement

In the first measurement, we restrict ourselves to a 2-dimensional Hilbert space, where we can define a Bloch sphere analogously to the one representing the polarisation of photons (Figure 3.13, left). The North and South Poles are helical IG modes; the equator represents a superposition with a well-defined phase-relation. The whole Bloch sphere can be represented by

\[
IG_{p,m,\epsilon}^\alpha(r) = \sqrt{\alpha} \cdot e^{i\phi} \cdot IG_{p,m,\epsilon}^+(r) + \sqrt{1-\alpha} \cdot e^{-i\phi} \cdot IG_{p,m,\epsilon}^-(r)
\]  

(3.37)

where \( \alpha \) goes from 0 to 1, and \( \phi \) goes from 0 to \( \pi \). The first two members of the Ince-Gauss family are equivalent to the first two modes of the Laguerre-Gauss family (namely \( IG_{0,0} = LG_{0,0} \) and \( IG_{1,1} = LG_{0,1} \)), therefore the first different mode is \( IG_{2,2,\epsilon>0} \) [11]. As a specific example, we analyse the \( IG_{p,m,\epsilon}^\alpha \) mode in a 2-dimensional subspace, which has two rings and three split vortices with an ellipticity \( \epsilon = 2 \). The hologram of a specific phase \( \phi \) is displayed at the SLM1, while the SLM2 scans through the holograms for phases from \( \phi = 0 \) to \( \phi = 180^\circ \). On both SLMs we display the phase-pattern for \( \alpha = 0.5 \), which corresponds to states at the equator of the Bloch sphere. In Figure 3.13, right the uncorrected coincidence counts are shown as a function of the phase of the hologram displayed at SLM2. We clearly see the non-classical fringes, with a very high visibility.

![Figure 3.13: Left: The Bloch sphere constructed from the Ince-Gauss modes IG\(_{5,3,2}\) with two rings and three vortices. The insets show the intensity (left) and phase patterns (right). Similar to the Laguerre-Gauss (LG) modes, the intensity patterns at the North and South poles are identical. However, in contrast to LG modes, where a continuous phase change of \( \phi \) along the equator only leads to a rotation of the phase and intensity pattern, these patterns also change their shape continuously, when going around the equator. Right: Coincidence-fringes for the Ince-Gauss IG\(_{5,3,2}\) mode with four different settings for the signal photon and 15° phase steps of the superposition for the idler photon using 60 mW of](image-url)
pump power. Each point has been measured for 15 seconds. We estimated the statistical uncertainty using Poisson distribution (the obtained error bars are smaller than the symbols in the figure). The lines show $\sin^2$-fits to the measured data.

The same method is shown for four other Ince-Gauss modes with a different degree and order parameter. These results (Figure 3.14) also show non-classical coincidence-fringes with a very high contrast, which indicates that the photons are highly entangled in the Ince-Gauss basis.

![Coincidence-fringes for different Ince-Gauss modes with ellipticity $\epsilon=2$. From top-left to bottom-right: IG$_{2,2,2}$, IG$_{3,3,2}$, IG$_{4,4,2}$, IG$_{4,2,2}$. The contrast of the fringes depends only on the phase difference. The first three have 2, 3 and 4 vortices and no additional rings and each data point has been measured for 5 seconds. The last one has two vortices and one additional ring. Each point has been measured for 15 seconds. The violet, green, red and blue points correspond to the measurement setting of the second Spatial Light Modulator SLM2 (22.5°, 67.5°, 112.5° and 157.5°). The obtained error bars from Poisson distribution are smaller than the symbols in the figure. The lines show $\sin^2$-fits.](image)
For quantifying the entanglement, we take advantage of an entanglement witness operator [47]. Similar to entangled OAM states from down-conversion, we expect a Bell state

$$|\Psi^+> = \frac{1}{\sqrt{2}} \left( |IG_{p,m,e}^+, IG_{p,m,e}^- > + |IG_{p,m,e}^-, IG_{p,m,e}^+ > \right) \tag{3.38}$$

of the down-converted photon pair, therefore a suitable witness operator for detecting entanglement in this state can be written as

$$\hat{W} = \frac{1}{4} \left( 1 - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z \right), \tag{3.39}$$

where $\sigma_x$, $\sigma_y$ and $\sigma_z$ denote the single-qubit Pauli matrices for the first and second photon. The Pauli matrices correspond to measurements in different mutually unbiased bases. The witness operator is defined in such a way, that it must be positive for all separable states, and will give $<W> = -0.5$ for the maximally entangled state. It is directly related to the fidelity with respect to the target state by $F = \frac{1}{2} - <W>$. The values of the entanglement witness calculated from our measurement results are given in Table 3.1.

<table>
<thead>
<tr>
<th>IG mode parameters</th>
<th>Witness $&lt;W&gt;$</th>
<th>Fidelity $F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p=2$, $m=2$, $\epsilon=2$</td>
<td>-0.4847(3)</td>
<td>98.47(3)</td>
</tr>
<tr>
<td>$p=3$, $m=3$, $\epsilon=2$</td>
<td>-0.4897(3)</td>
<td>98.97(3)</td>
</tr>
<tr>
<td>$p=4$, $m=4$, $\epsilon=2$</td>
<td>-0.4905(4)</td>
<td>99.05(4)</td>
</tr>
<tr>
<td>$p=4$, $m=2$, $\epsilon=2$</td>
<td>-0.4581(7)</td>
<td>95.81(7)</td>
</tr>
<tr>
<td>$p=5$, $m=3$, $\epsilon=2$</td>
<td>-0.4784(7)</td>
<td>97.84(7)</td>
</tr>
</tbody>
</table>

Table 3.1: For five different IG modes we have calculated the witness as described in equation (8), and the corresponding fidelity to the target state. The negative witness proves entanglement of our states and the fidelity shows that the measured state from the down-conversion is close to the expected one. The statistical uncertainty given in brackets has been calculated assuming Poisson distributed statistics.

The witness $<W>$ for every measured mode is negative, which verifies entanglement of the generated states.

### 3.4.2. 3-dimensional entanglement

We prove that the photons are indeed genuinely entangled in a higher-dimensional Hilbert space. For this task, we use the first non-trivial 3-dimensional Ince-Gauss space. Such a state can be written as

$$|\Psi> = a|IG_{2,2,5}^+, IG_{2,2,5}^- > + b|IG_{3,3,5}^+, IG_{3,3,5}^- > + c|IG_{4,4,5}^+, IG_{4,4,5}^- >, \tag{3.40}$$

where $a$, $b$ and $c$ are weighting constants, for example due to unequal generation rates of different Ince-Gauss modes in the SPDC process. An analogous effect also exists for Laguerre-Gauss modes [48][49][50].
Similarly to the 2-dimensional case, we can define an entanglement witness for three dimensions, which consists of the visibilities in three mutually unbiased bases for every 2-dimensional subspace [51][52]. The simplest correlation function of this type can be written as
\[ f(\rho) = \sum_{l=3}^{4} \sum_{k<l} \left( \langle \sigma_x^{k,l} \otimes \sigma_x^{k,l} \rangle + \langle \sigma_y^{k,l} \otimes \sigma_y^{k,l} \rangle + \langle \sigma_z^{k,l} \otimes \sigma_z^{k,l} \rangle \right). \] 

The \( \sigma_{\pm i} \) is a Pauli matrix constructed from \( \text{IG}^\pm \) and denotes the measurements in the mutually unbiased bases of a 2-dimensional subspace of the 3-dimensional state. \( N_{k,l} \) are normalisation constants that appear because we ignore the third degree of freedom in the 2-dimensional measurement.

Table 3.2: Visibilities in the three mutually unbiased bases of the three 2-dimensional subsystems (basis vectors are given in the first row of the table). The measurement time for each data point was 5 seconds and the statistical error has been calculated assuming Poisson distributed counts.

<table>
<thead>
<tr>
<th></th>
<th>(\text{IG}<em>{2,2.5}, \text{IG}</em>{3,3.5})</th>
<th>(\text{IG}<em>{2,2.5}, \text{IG}</em>{4,4.5})</th>
<th>(\text{IG}<em>{3,3.5}, \text{IG}</em>{4,4.5})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_x \otimes \sigma_{-x} )</td>
<td>0.917(1)</td>
<td>0.932(1)</td>
<td>0.890(2)</td>
</tr>
<tr>
<td>( \sigma_y \otimes \sigma_{-y} )</td>
<td>0.923(1)</td>
<td>0.834(2)</td>
<td>0.872(2)</td>
</tr>
<tr>
<td>( \sigma_z \otimes \sigma_{-z} )</td>
<td>0.938(1)</td>
<td>0.945(1)</td>
<td>0.905(2)</td>
</tr>
</tbody>
</table>

The bounds for the 3-dimensional entanglement witness (3.41) has been calculated by Marcus Huber, the derivation can be seen in the Supplementary of our paper [53] and in the appendix (B).

The results are:

1. Limit for separable states: \( f(\rho) = 3 \)
2. Limit for 2-dimensionally entangled states: \( f(\rho) = 6 \)
3. Overall maximum: \( f(\rho) = 9 \)

The measured visibilities in (Table 3.2) combined with the correlation function in equation (3.41) give the value
\[ f(\rho) = 8.156(5), \] 
which is well above the limit for an entangled state in two dimensions, and therefore verifies that the measured state was at least a 3-dimensionally entangled state.

3.4.3. \( \epsilon \)-dependent overlap

The ellipticity is a unique feature of IG modes, which does not exist for LG or HG modes. This is a novel resource for quantum information processing. We measure the correlation between Ince-Gauss modes with the same number of rings and singularities, but with different ellipticity.

On the SLM1 we display an IG mode with a specific ellipticity, and on the SLM2 we display a mode with the same characteristic numbers \( p \) and \( m \) but different \( \epsilon \). When the
two $\epsilon$ match, we measure a maximum coincidence count rate, for different ellipticities $\epsilon$ and $\epsilon'$ the coincidences decrease. The theoretical calculated overlap can be seen in Figure 3.15.

Figure 3.15: Theoretically calculated overlap between modes with the same characterizing quantum numbers, but different ellipticity $\epsilon$. The x- and y-axis correspond to $\epsilon$ and $\epsilon'$, while the z-axis shows the overlap between the two modes. The plots have been created by extending a Mathematica program by William Plick, which calculates in an analytical way the decomposition of Ince-Gauss into Laguerre-Gauss modes. Two important features can be seen: firstly, the overlap is one for $\epsilon=\epsilon'$, and decreases for different values of ellipticity. Secondly, the third plot shows that the overlap reaches zero and increases again (in fact, the amplitude crosses zero and becomes negative). This suggests that there are orthogonal modes with same $p$ and $m$ number, but different $\epsilon$ and $\epsilon'$.

The decrease of the coincidence rate is bigger for higher-order modes (as it can be seen in figure 3.15), therefore we used $\text{IG}_{8,4,\epsilon}$, which is still accessible in the down-conversion without long measurement times. The calculated overlap (where we assume a maximally entangled state) and measured coincidence counts are shown in figure 3.16.

Figure 3.16: Overlap between two Ince-Gauss modes with the same value of $p$ and $m$, but different ellipticities. This measurement shows a unique behaviour of Ince-Gauss modes,
which might be useful for quantum mechanical applications. The horizontal axis is the ellipticity; the vertical axis shows the coincidence counts. The blue line shows the theoretical overlap $| < IG_{8,4,3}|IG_{8,4,4}|^2$, where the maximal overlap is at $\epsilon = 3$. The red dots are measured coincidence counts, which have a very similar behaviour as the calculated values. We surmise that the small deviation from the theoretical value is due to the unequal creation probability for different modes (which we have not accounted for in the theory curve) in the down-conversion process, and possibly also from higher order mode effects in the projection process. These effects could be compensated by mode-specific Procrustean filtering [54] and Gauss-mode holograms [45]. The obtained error bars are smaller than the symbols in the figure.

### 3.4.4. Vortex correlations

The splitting of the vortices and their corresponding singluarities in Ince-Gauss modes is one of their interesting features. One could cut out individual vortices at each SLM, and look at the correlations between those vortices. Figure 3.17 shows how the vortex are cut out in the phase mask of the SLM for a specific mode.

![Figure 3.17](image)

**Figure 3.17:** Phase-pattern of an Ince-Gauss mode with suggested cut out vortices. The bright circles represent the hologram that is applied – which is not cut out.

At each SLM in the experiment, one specific vortex is displayed. The remaining part of the hologram does not get a phase-shift. Due to the grating that is used to separate different diffraction orders, this method is similar to a circular apperture in the beam.
Figure 3.18: Correlations detected when at each SLM one specific vortex is cut out. In the measurement, IG$_{3,3,5}$ and IG$_{4,4,5}$ have been used with 3 and 4 vortices respectively. High coincidence counts are detected when the opposite vortices are used to measure, low coincidences for other vortices.

Figure 3.18 shows that high coincidences are detected when opposite vortices are displayed on the two SLMs. It seems like this effect could be explained in a momentum-entanglement picture.

As an extension to this measurement for getting new insights into the topological charge of IG modes, one could measure coincidences of cut out vortices with other phase structures.
4. Maverick beams

4.1. Construction of Maverick beams

Maverick beams are artificially constructed beams extending in the first instance the Laguerre-Gauss modes. As it can be seen in Figure 2.2, Laguerre-Gauss modes have a simple well-defined phase structure

(I) in angular direction, their phase changes linearly from 0 to $2\pi \cdot l$, with $l \in \mathbb{Z}$

(II) in radial direction, the phase acquires an additional $\pi$-phase at every root of the Laguerre polynomials. This effectively defines a ring-structure in the phase distribution.

Maverick beams are constructed in a well-defined way to remove these restrictions, with Laguerre-Gauss modes as special case:

1. in every ring $r_i$, the angular phase goes from 0 to $2\pi l_i$, $l_i \in \mathbb{Z}$ (special case LG: $\forall l_i : l_i = l$)

2. every radial ring acquires an additional phase $\Delta_i$ between 0 and $2\pi$ (special case LG: $\forall \Delta_i : \Delta_i = (i - 1) \cdot \pi$)

There are several other ways to extend this set. However, just by implementing these rules to define artificial phase pattern opens up an immense amount of possibilities.

In figure (4.1), one can see the phase pattern created by using the additional rule 1 and the measured intensity distribution, while in figure (4.2) the phase pattern and intensity distribution for rule 2 is shown.

The Maverick phase patterns for $n$ radial segments can be written compactly as

$$mav := mav_{l_1, \Delta_1, \ldots, l_n, \Delta_n} := mav \otimes_{i=1}^{n} l_i, \Delta_i$$

(4.1)

![Figure 4.1: Maverick beams with constant $\Delta_i = 0$, and varying $l_1, l_2$.](image)
4.2 Quantum mechanical properties

4.2.1. Measuring of qubit correlations

To analyze qubit entanglement, the definition of a Bloch sphere is important. As the Maverick beams are artificially constructed and not mathematically investigated yet, analytical results for orthogonality do not exist.

However, we can try to define a Bloch sphere of Maverick beams analog to the Bloch sphere of Ince-Gauss modes (Figure 3.13). For Ince-Gauss modes the phase pattern of the south pole is \(2\pi-(\text{phase of north pole})\). The same rule can be applied to Maverick beams, and in analogy to equation (3.37) for Ince-Gauss modes, the Bloch sphere of Maverick beams can be written as

\[
\text{mav}^{a, \phi}_{\otimes_{i=1}^{n} l_i, \Delta_i} := \left( \sqrt{a} \cdot e^{i\phi} \cdot \text{mav}_{\otimes_{i=1}^{n} l_i, \Delta_i} + \sqrt{1-a} \cdot e^{-i\phi} \cdot (2\pi - \text{mav}_{\otimes_{i=1}^{n} l_i, \Delta_i}) \right) \tag{4.2}
\]
4.2 Quantum mechanical properties

Figure 4.3: Bloch sphere of $\text{mav}_{1,0,-2,0}$, defined in the same way as for Ince-Gauss modes.

We perform the same measurement that we did for Ince-Gauss, using the experimental setup of Figure 3.9. As before, the hologram with a specific phase $\phi$ is displayed at the SLM1, while the SLM2 scans through the holograms with phases from $\phi=0^\circ$ to $\phi=180^\circ$. On both SLMs we display the phase-pattern for $a=0.5$ - the states at the equator of the Bloch sphere. In Figure 4.4 the uncorrected coincidence counts are shown as a function of the phase of the hologram displayed at SLM2. We clearly see the non-classical fringes.

Figure 4.4: Dependence of coincidence rate on the phase for three different Maverick modes: $\text{mav}_{1,0,2,0}$, $\text{mav}_{1,0,-2,0}$ and $\text{mav}_{1,0,1,90,1,0}$. The datapoint of the first two figures have been each measured for 15 seconds, the data points in the last figure each for 5 seconds. We estimated the statistical uncertainty using Poisson distribution (the obtained error bars are smaller than the symbols in the figure). The lines show $\sin^2$-fits to the measured data.
4.2 Quantum mechanical properties

4.2.2. Extensions

One can extend the Maverick beams further; one possibility is the following rule:

3. the angular phase change can be given by a nonlinear function \( f_i(\phi) \) (special case LG: 
\[ \forall f_i(\phi) : f_i(\phi) = l \cdot \phi \] 

We tried the same approach to find orthogonal modes, namely presume that modes with an inverse phase structure are orthogonal. The function we used was \( f(\phi) = 2\pi \cos(l \cdot \phi) \), \( f(\phi) = \tan(l \cdot \phi) \) and \( f(\phi) = \sinh(l \cdot \phi) \), Results are shown in Figure 4.5 - 4.7.

\[ 2\pi \cos(1\phi) \quad 2\pi \cos(2\phi) \quad 2\pi \cos(3\phi) \quad 2\pi \cos(4\phi) \quad 2\pi \cos(5\phi) \]

\[ \text{Figure 4.5: Coincidence counts for Maverick Beams with cosine-like phase dependence, with different frequencies } l=\{1,2,3,4,5\}. "+" \text{ stands for a given Maverick beam, "}-" \text{ stands for its inverse (2\pi-phase pattern). The coincidence counts for } l=1 \text{ have been divided by } 4, \text{ to make the figure clearer. One can see that it depends on the symmetry of the phase pattern whether inverse phase leads to correlation or anticorrelation. For vertically mirror-symmetric modes we detect correlation between the mode and its inverse one. If this symmetry is not present, there is an anticorrelation for pairwise inverse modes, but correlation for the same phase pattern.} \]
4.2 Quantum mechanical properties

Figure 4.6: Coincidence counts for Maverick Beams with tangent-like phase dependence, with different frequencies \( l=\{1,2,3,4,5\} \). + stands for a given Maverick beam, - stands for its inverse (2\(\pi\)-phase pattern). For small \( l \), there is clearly correlation between pairwise inverse modes. For bigger \( l \), this correlation vanishes. In contrast to cosine-like phase dependence, there is a much bigger coincidence rate for modes with the same phase structure. One can presume that both effects come from singular regions in the phase, where the tangent gets very big.

Figure 4.7: Coincidence counts for Maverick Beams with sinh-like phase dependence, with different frequencies \( l=\{1/3, 2/3, 1\} \). + stands for a given Maverick beam, - stands for its inverse (2\(\pi\)-phase pattern). The coincidence counts for \( l=1/3 \) have been divided by 10, to make the figure more suggestive. For \( l=1/3 \), correlation between pairwise inverse modes is...
detected, for \( l = \frac{2}{3} \) there is anticorrelation, and high number of coincidences for modes with the same hologram. For \( l = \frac{3}{3} \), coincidences nearly vanish for the same phase patterns as well as for inverse ones. One can speculate that this is due to the very fast increase of the hyperbolic sine with the resulting increasing number of phase jumps in the angular direction.

Another possibility to extent Maverick beams is the implementation of the following rule:

4. the parameters \( l_i \) can be a real number \( l_i \in \mathbb{R} \) (special case LG: \( l_i = l \in \mathbb{Z} \))

With \( f_i(\phi) = l \cdot \phi \), this is the special case of fractional OAM beams, as described in chapter 2.3.1. We look at coincidences between inverse phase-patterns again, and measure coincidence counts. The results can be seen in Figure 4.8.

![Figure 4.8](image)

**Figure 4.8**: Coincidence counts for pairwise inverse Maverick Beams with \( l \in \mathbb{R} \), with \( l=[1.0, 1.25, 1.4, 1.45, 1.5, 1.75, 2] \). The points represent measured values, the lines are interpolations between the points. We discover that there is a minimum of coincidences at \( l=1.5 \), and maxima in the coincidence rate at \( l=1 \) and \( l=2 \) - which are the original Laguerre-Gauss modes. From the decomposition of fractional OAM to Laguerre-Gauss modes

In the end we tried to analyse random patterns. We did not set every pixel of the SLM to a random value but restricted ourselves to small squares with a phase of \( \phi \) from 0 to \( 2\pi \). The results can be seen in Figure 4.9.
4.2 Quantum mechanical properties

Figure 4.9: Coincidence counts random square phase patterns. The + stands for a given Maverick beam, - stands for its inverse (2\(\pi\)-phase pattern). The coincidence counts of the second phase pattern have been divided by 4, to make the figure more suggestive.

No general behaviour of the coincidence counts can be observed which suggests that the inverse-phase pattern method is not a general way to create orthogonal modes. This becomes obvious, when one looks at the equator of Ince-Gauss Bloch spheres (Figure 3.13). The orthogonal modes in this case have a different structure, not just an inverse phase. A suggestive example would a Hermite-Gauss mode, which is shown in Figure 4.10.

Figure 4.10: The inverse-phase pattern method applied to Hermite-Gauss (HG) modes. In the left a Hermite-Gauss phase pattern is shown. The middle picture has been created via (2\(\pi\) - HG ). It turns out that this operation just adds a global phase to the pattern, such that it physically exactly the same phase pattern.

A general method for obtaining orthogonal modes has yet to be found before using the extended Maverick beams in quantum mechanical experiments.
5. Conclusion and Outlook

In this thesis, we analysed the spatial structure of photons beyond the well-known and widely-used Laguerre-Gauss modes.

The first part covers the Ince-Gauss modes. We analysed in a classical way the vortices and singularity structure of these modes. Further investigations into the behaviour of the singularities as well as the corresponding topological charge seems to be possible based on the method shown in chapter 3.2.1.

The entanglement of Ince-Gauss modes has been shown in two and three dimensions. In addition, an Ince-Gauss specific quantum correlation function has been measured which might be used as a novel quantum resource. This part has been combined to a research paper [53] and submitted to a scientific journal. In appendix A, a simple idea for an enhanced quantum key distribution protocol using Ince-Gauss modes is described.

The second part covers a novel family of artificial modes – the **Maverick beams**. We have defined, created and analysed them for the first time. The idea in this approach was to create artificial phase structures based on Laguerre-Gauss modes by adding new degrees of freedom to the original pattern. It seems possible (but has not yet been proven) that some of the resulting phase patterns do not represent the phase of a solution of the paraxial wave equation. The imaged modes created by these phase structures give fascinating results, such as spiral shapes or smooth generation of additional radial rings in the intensity profile.

The entanglement of three members of Maverick beams has been shown. This was possible by defining an artificial Bloch sphere in an analog way as for Ince-Gauss modes. Due to the additional degrees of freedom, an enormous set of possible Maverick modes can be created. This opens up a broad topic for future research, in a classical as well as in a quantum mechanical way – including the search for novel applications in quantum information protocols.
A. Quantum Cryptography with Ince-Gauss modes

For Ince-Gauss modes, the additional parameter $\epsilon$ might be useful for quantum cryptography. One very simple approach could be the following extension to BB84 [55] protocol. The extension to the Ekert91 [56] protocols is straight forward.

1. First a bitstream is transmitted in the standard way of the protocol. However, instead of using this bitstream as key, it is transmitted to generate a random ellipticity $\epsilon \in \mathbb{R}^+$. 

2. This ellipticity $\epsilon$ defined the basis set of the real key distribution, which will be performed in a second step.

The advantage is that an eavesdropper Eve cannot know the ellipticity $\epsilon$ exactly, otherwise her presence would be detected in step 1. As she does not know the ellipticity exactly, she will introduce additional errors, even if she guesses the basis right.

This is due to the decreasing overlap of Ince-Gauss modes with same characteristic numbers $p$ and $m$, but different ellipticity $\epsilon$, as it has been shown in chapter 3.4.3.

Changing the ellipticity $\epsilon$ is not a the same type of rotation of the basis vectors as it would be possible with polarisation. Rather changing the ellipticity is a rotation of the infinite-dimensional basis set of the Hilbert space. The infinite-dimensional Ince-Gauss basis set is complete. Lower-dimensional subsets are not complete. When we restrict ourselves to a 2-dimensional subspace and change $\epsilon$, we cant reconstruct the old basis with the new one.

This is in contrast to rotation of the polarisation basis, where one can reconstruct the old basis by a superposition with the new basis.

The extended QKD is not possible with Laguerre-Gauss or Hermite Gauss modes, as the continuous parameter for high-dimensional rotation is missing. However, it should be mentioned that this method is not restricted to Ince-Gauss modes, but would work for any family of modes that has a continuous parameter defining a different basis for each value, and a higher-dimensional Hilbert space that has subsets of the basis which are in themselves not complete.


B. Bounds of the high-dimensional correlation function

This chapter is the supplementary information for our paper [53]. The calculation has been done by Marcus Huber in order to provide the bounds for the correlation function, which we use to verify the presence of three dimensionally entangled modes in chapter 3.4.2. For completeness, the proof is also included here.

B.1. Proof that $f(\rho) \leq 6$ for two-dimensionally entangled states

Using the following abbreviations

$$
|IG^+_{2,3,5}\rangle = |0_+\rangle, |IG^+_{3,3,5}\rangle = |1_+\rangle, |IG^+_{4,4,5}\rangle = |2_+\rangle,
$$

$$
|IG^-_{2,3,5}\rangle = |0_-\rangle, |IG^-_{3,3,5}\rangle = |1_-\rangle, |IG^-_{4,4,5}\rangle = |2_-\rangle,
$$

we can represent the performed measurements via the following operators

$$
\sigma_{x}^{kl \pm } := |k\pm \rangle\langle k\pm | + |l\pm \rangle\langle l\pm |
$$

$$
\sigma_{y}^{kl \pm } := i|k\pm \rangle\langle k\pm | - i|l\pm \rangle\langle l\pm |
$$

$$
\sigma_{z}^{kl \pm } := |k\pm \rangle\langle k\pm | - |l\pm \rangle\langle l\pm |.
$$

In order to lower bound the dimensionality of entanglement we use the following correlation function

$$
f(\rho) = g(\rho^{01}) + g(\rho^{02}) + g(\rho^{12}),
$$

where $\rho^{kl}$ are the normalized subspace density matrices, where all but two degrees of freedom on both sides are ignored, i.e.

$$
\rho^{kl} := \frac{(|k_+\rangle\langle k_+ | + |l_+\rangle\langle l_+ |) \otimes (|k_-\rangle\langle k_- | + |l_-\rangle\langle l_- |) \rho (|k_+\rangle\langle k_+ | + |l_+\rangle\langle l_+ |) \otimes (|k_-\rangle\langle k_- | + |l_-\rangle\langle l_- |)}{N_{kl}},
$$

where $N_{kl}$ is the normalization, such that $\text{Tr}(\rho^{kl}) = 1$, and

$$
g(\rho^{kl}) = \text{Tr} \left( (\sigma_{+z}^{kl} \otimes \sigma_{-z}^{kl} - \sigma_{+y}^{kl} \otimes \sigma_{-y}^{kl} + \sigma_{+x}^{kl} \otimes \sigma_{-x}^{kl}) \rho^{kl} \right).
$$

Comparing these to the correlations on the total state, i.e.

$$
f_{kl} = \text{Tr} \left( (\sigma_{+z}^{kl} \otimes \sigma_{-z}^{kl} - \sigma_{+y}^{kl} \otimes \sigma_{-y}^{kl} + \sigma_{+x}^{kl} \otimes \sigma_{-x}^{kl}) \rho \right),
$$
B.1 Proof that $f(\rho) \leq 6$ for two-dimensionally entangled states

we can write

$$g(\rho_{kl}) = \frac{f_{kl}}{N_{kl}} ,$$

and thus

$$f(\rho) = \sum_{k<l} \frac{f_{kl}}{N_{kl}} .$$

A bound can easily be obtained for the convex function as it is maximized by pure states. We just need to optimize over pure states with Schmidt-Rank two, in order to get a bound for two dimensionally entangled states, i.e.

$$\sum_{k<l} f_{kl} \leq \max_{|\psi_2\rangle} f(|\psi_2\rangle\langle\psi_2|) ,$$

where $|\psi_2\rangle = \lambda_1 |v_1 v_1^\prime\rangle + \lambda_2 |v_2 v_2^\prime\rangle$. Lagrangian maximization leads to three distinct maximizing states

- $|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0,0\rangle + |1,1\rangle)$
- $|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0,0\rangle + |2,2\rangle)$
- $|\psi_2\rangle = \frac{1}{\sqrt{2}} (|1,1\rangle + |2,2\rangle)$.

Then using the fact that $\Re \{\langle\alpha|\rho|\beta\rangle\} \leq \frac{1}{2} (\langle\alpha|\rho|\alpha\rangle + \langle\beta|\rho|\beta\rangle)$ (due to the positivity of the density matrix) we can bound the expression via

$$\sum_{k<l} f_{kl} \leq 4 \left( \langle 0,0|0,0\rangle + \langle 1,1|1,1\rangle + \langle 2,2|2,2\rangle \right) .$$

Now we are left with the following Lagrangian problem

$$f(\rho, \lambda) = \sum_{k<l} \frac{f_{kl}}{N_{kl}} + \lambda \left( \sum_{k<l} f_{kl} - 2 \left( \sum_{k=0}^{2} \langle k,k-|\rho|k,k-\rangle \right) \right) ,$$

which we want to solve in order to maximize $f(\rho)$. Taking

$$\frac{\partial}{\partial \Re \{\langle k+k-|\rho|l+l-\rangle\}} f(\rho) = \frac{1}{N_{kl}} + \lambda = 0 ,$$

shows that every extremal point has to satisfy

$$\lambda = -\frac{1}{N_{01}} = -\frac{1}{N_{02}} = -\frac{1}{N_{12}}$$

$$\Rightarrow N_{01} = N_{02} = N_{12} =: N .$$

In consequence for any extremal point we can directly bound

$$f(\rho) \leq \sum_{k<l} \frac{f_{kl}}{N} - \frac{1}{N} \left( \sum_{k<l} f_{kl} - 2 \left( \sum_{k=0}^{2} \langle k,k-|\rho|k,k-\rangle \right) \right) = \frac{4 \left( \sum_{k=0}^{2} \langle k+k-|\rho|k,k-\rangle \right)}{N} .$$

Now we can use that

$$N = \frac{1}{3} (N_{01} + N_{02} + N_{12})$$
and have a lower bound on $N$, which directly leads to another lower bound on $f(\rho)$:

$$f(\rho) \leq \frac{4(\sum_{k=0}^{2}(k+1)\rho|k+k, k)_k)}{2(\sum_{k=0}^{2}(k+1)\rho|k+k, k)_k)} = 6. \tag{B17}$$

Surprisingly this bound is tight, i.e. it can be saturated by the following two dimensionally entangled state

$$\rho_2 = \frac{1}{3}(|\phi_{01}^+\rangle|\phi_{02}^+\rangle + |\phi_{02}^+\rangle|\phi_{12}^+\rangle), \tag{B18}$$

where $|\phi_{kl}^+\rangle := \frac{1}{\sqrt{2}}(|k, k, +\rangle + |l, l, -\rangle)$, where $f(\rho_2) = 6$. This is intriguing as $f(|\phi_{02}^+\rangle|\phi_{12}^+\rangle) = 5$, meaning that we have indeed bounded a non-convex entanglement detection criterion.

### B.2. Proof that $f(\rho) \leq 3$ for separable states

This follows from the fact that every $2 \times 2$ dimensional subspace of a separable state is again separable. Thus

$$g(\rho^{kl}) = (\sigma_{+z}^{kl} \otimes \sigma_{-z}^{kl}) - (\sigma_{+y}^{kl} \otimes \sigma_{-y}^{kl}) + (\sigma_{+x}^{kl} \otimes \sigma_{-x}^{kl}) \tag{B19}$$

$$= (\sigma_{+z}^{kl}) \cdot (\sigma_{-z}^{kl}) - (\sigma_{+y}^{kl}) \cdot (\sigma_{-y}^{kl}) + (\sigma_{+x}^{kl}) \cdot (\sigma_{-x}^{kl}) \tag{B20}$$

$$= \begin{pmatrix} (\sigma_{+z}^{kl}) & (\sigma_{+y}^{kl}) \\ (\sigma_{+y}^{kl}) & (\sigma_{+z}^{kl}) \end{pmatrix} \cdot \begin{pmatrix} (\sigma_{-z}^{kl}) \\ (\sigma_{-z}^{kl}) \end{pmatrix} \leq \begin{pmatrix} (\sigma_{+z}^{kl}) \\ (\sigma_{+y}^{kl}) \end{pmatrix} \cdot \begin{pmatrix} (\sigma_{-z}^{kl}) \\ (\sigma_{-y}^{kl}) \end{pmatrix} \leq 1, \tag{B21}$$

due to the fact that the local Bloch vectors’s length is limited by one. As

$$f(\rho) = g(\rho^{01}) + g(\rho^{02}) + g(\rho^{12}) \leq 3, \tag{B22}$$

we have completed the proof. Also this bound is tight as e.g. for $\rho_S = \frac{1}{3}(|0, 0\rangle|0, 0\rangle + |1, 1\rangle|1, 1\rangle + |2, 2\rangle|2, 2\rangle)$, it is easy to see that $f(\rho_S) = 3$. 
C. Source codes

C.1. Ince-Gauss calculations

```matlab
% fctInceVortex.m
x00=0; y00=0;
tmpFind=findstr(C,',');
tmpP=0;
tmpM=0;
if(size(tmpFind,2)~=[3 && size(tmpFind,2)~=[4])
    fprintf('Wrong format of Ince-Gaussian input. Should have the form: ...
           ivr,#,#,e(,I)
           Where r is rotation-direction (0 for +, 1 for -), # is an integer ...
           number and e=eccentricity, I is intensity');
else
    tmpP=real(str2double(C(tmpFind(1):tmpFind(2)-1)));
    tmpM=real(str2double(C(tmpFind(2):tmpFind(3)-1)));
    if (size(tmpFind,2)==3)
        elips=real(str2double(C(tmpFind(3):end)));
        DrawInt=0;
    else
        elips=real(str2double(C(tmpFind(3):tmpFind(4))));
        DrawInt=1;
        if (exist(['savevalue/ig/iv',num2str(nx), ',', num2str(ny), ',', ...
                    num2str(w0), ',', num2str(tmpRot), ',', num2str(tmpP), ',', ...
                    num2str(tmpM), ',', num2str(elips)],'file'))
            HelicialInceFct=dlmread(['savevalue/ig/iv',num2str(nx), ',', ...
                                    num2str(w0), ',', num2str(tmpRot), ',', ...
                                    num2str(tmpP), ',', num2str(tmpM), ',', num2str(elips)]);
        else
            if (sum((sum(u.*u))==0) || (oldelips~elips || oldw0~ w0))
                if (exist(['savevalue/ig/elu', num2str(nx), ',', num2str(ny), ',', num2str(elsips)],'file') && ...
                    exist(['savevalue/ig/elv', num2str(nx), ',', num2str(ny), ',', num2str(elsips)],'file'))
                    u=dlmread(['savevalue/ig/elu',num2str(nx), ',', num2str(ny), ... 
                               ', num2str(w0), ',', num2str(elsips)]);
                    v=dlmread(['savevalue/ig/elv',num2str(nx), ',', num2str(ny), ...
                               ', num2str(w0), ',', num2str(elsips)]);
                else
                    fprintf('Now creating the elliptical coordinates for the new ...
                           ellipticity parameter, this may take some time...

                           [u v]=fctCartesianToElliptic(X,Y,w0*sqrt(elips/2));
```
C.1 Ince-Gauss calculations

```matlab
30 dlmwrite(['savevalue/ig/elv', num2str(nx), ',', num2str(ny), '], u, 'precision', 10);
31 dlmwrite(['savevalue/ig/elv', num2str(nx), ',', num2str(ny), '], v, 'precision', 10);
32 end
33 oldelips = elips;
34 oldw0 = w0;
35 end
36
37 fprintf('Calculating koefficients for IG polynoms\n
38 rr^2 = w0^2 * (elips/2) * ((cosh(u).*cos(v))^2 + (sinh(u).*sin(v))^2);
39 PreFakt = exp(-rr^2/(w0^2));
40 coeffe = fctSolveRecurrenceRelation('e', tmpP, tmpM, elips);
41 IGe = PreFakt .* fctIncePolyVal('e', tmpP, tmpM, elips, 1i*u, coeffe) .* ...
42 fctIncePolyVal('e', tmpP, tmpM, elips, v, coeffe);
43 coeffo = fctSolveRecurrenceRelation('o', tmpP, tmpM, elips);
44 IGo = PreFakt .* fctIncePolyVal('o', tmpP, tmpM, elips, 1i*u, coeffo) .* ...
45 fctIncePolyVal('o', tmpP, tmpM, elips, v, coeffo);
46
47 r = [0:(tmpP-mod(tmpP,2))/2]; % Basis as function (for fast numerical ... integration)
48 if (mod(tmpP,2)==0)
49   % disp('sin(2 *r*x) && cos(2*r*x)');
50   be = @(argx) cos(2 * kron(r,argx(:)));
51   bo = @(argx) sin(2 * kron(r,argx(:)));
52 else
53   % disp('sin(((2 *r)+1)*x) && cos(((2*r)+1)*x)');
54   be = @(argx) cos(2 * kron(r,argx(:))+ argx(:)*ones(1,size(r,2)));
55   bo = @(argx) sin(2 * kron(r,argx(:))+ argx(:)*ones(1,size(r,2)));
56 end
57
58 fprintf('Normalisation...\n
59 Fe = @(argx) sum(bsxfun(@times, coeffe', be(argx),2)); % ...
60 coeffe(0)*cos(2*0*x) + coeffe(1)*cos(2*1*x) + coeffe(2)*cos(2*2*x) ... + ...
61 Fo = @(argx) sum(bsxfun(@times, coeffo', bo(argx),2)); % ...
62 coeffo(0)*sin(2*0*x) + coeffo(1)*sin(2*1*x) + coeffo(2)*sin(2*2*x) ... + ...
63
64 rrr^2 = @(argu,argv) w0^2 * (elips/2) * ((cosh(argu).*cos(argv))^2 ... + (sinh(argu).*sin(argv))^2); % r^2 in elliptic coordinates
65 PreFak^2 = @(argu,argv) exp(-2 * rrr^2/(w0^2)); % The Gaussian Part ...
66
67 IGe^2fc = @(argu,argv) PreFak^2(argu,argv) .* Fe(li*argu).*Fe(argv) .* ...
68 conj(Fe(li*argu).*Fe(argv)); % even IG^2 function
69 IGo^2fc = @(argu,argv) PreFak^2(argu,argv) .* Fo(li*argu).*Fo(argv) .* ...
70 conj(Fo(li*argu).*Fo(argv)); % odd IG^2 function
71
72 IGE2 = abs(IGe).^2;
73 IGO2 = abs(IGO)^2;
74 SE = sum(sum(IGE2));
75 SO = sum(sum(IGO2));
76
77 NormE = sqrt(SO/SE);
```
C.1 Ince-Gauss calculations

NormO=1;

if (tmpRot==0)
    HelicialInceFct=NormE*IGe + NormO*IGo;
elseif (tmpRot==1)
    HelicialInceFct=NormE*IGe - NormO*IGo;
else
    disp('Rotation has to be 0 or 1');
end
dlmwrite(['savevalue/ig/iv', num2str(nx), ',', num2str(ny), ',', ...
    num2str(w0), ',', num2str(tmpRot), ',', ...
    num2str(tmpP), ',', num2str(tmpM), ',', ...
    num2str(elips)], HelicialInceFct, 'precision', 5);
end

if (tmpRot==0 || tmpRot==1)
    if (DrawInt==0)
        IGintensity0=abs(HelicialInceFct).^2;
        IGintensity0=IGintensity0/(max(max(IGintensity0))) ;
        holo(gcf).data=angle(HelicialInceFct)+pi;
        figure(3)
        imagesc(angle(HelicialInceFct)+pi)
        figure(4)
        IGintensity0=abs(HelicialInceFct).^2;
        IGintensity0=IGintensity0*2*pi/(max(max(IGintensity0))) ;
        imagesc(IGintensity0)
        figure(1)
    else
        IGintensity0=abs(HelicialInceFct).^2;
        IGintensity0=IGintensity0*2*pi/(max(max(IGintensity0))) ;
        holo(gcf).data=IGintensity0;
        figure(3)
        imagesc(IGintensity0);
        figure(1)
    end
end

holo(gcf).mode=['I vortex' C(2:tmpFind(2)-1) tmpRot ' eccentricity ' ...
    C(tmpFind(2):end) ' Ince-Gaussian Beam.']; %updating mode info

%---
% Here we calculate DL of the beam
rsts=max(max(Rad))/(nx);
phists=2*pi/ny;
R=rsts:rsts:max(max(Rad));
P=phists:phists:max(max(Rad));
if (exist(['savevalue/ig/d',num2str(nx), ',', num2str(ny), ',', ...
    num2str(tmpRot), ',', num2str(tmpP), ',', num2str(tmpM), ',', ...
    num2str(elips), 'file']) && exist(['savevalue/ig/int',num2str(nx), ',', num2str(ny), ',', ...
    num2str(tmpRot), ',', num2str(tmpP), ',', num2str(tmpM), ',', ...
    num2str(elips), 'file'])
    ddif=dlmread(['savevalue/ig/d',num2str(nx), ',', num2str(ny), ',', ...
        num2str(tmpRot), ',', num2str(tmpP), ',', num2str(tmpM), ',', ...
        num2str(elips)]);
    intensity=dlmread(['savevalue/ig/int',num2str(nx), ',', num2str(ny), ...
        ',', num2str(tmpRot), ',', num2str(tmpP), ',', num2str(tmpM), ',', ...
        num2str(elips)]);
C.1 Ince-Gauss calculations

```matlab
else
disp('Calculate r/phi elliptic coordinates :-|');
[un,vn]=fctPolarToElliptic(R,P,x00,y00,w0*sqrt(elips/2));
disp('Calculate HIG in r/phi coordinates');

\[ rr_2 = w_0^2 \times (elips/2) \times ((cosh(un) \times cos(vn))^2 + (sinh(un) \times sin(vn))^2) \];
PreFakt=exp(-rr_2/(w0*w0));

IGeP1=0; IGeP2=0; IGoP1=0; IGoP2=0;
if (mod(tmpP,2)==0)
    for i=1:numel(coeffe)
        IGeP1=IGeP1 + (coeffe(i)*cosh(2*(i-1)*un));
        IGeP2=IGeP2 + (coeffe(i)*cos(2*(i-1)*vn));
    end
else
    for i=1:numel(coeffe)
        IGeP1=IGeP1 + (coeffe(i)*cosh(2*(i)*un));
        IGeP2=IGeP2 + (coeffe(i)*cos(2*(i)*vn));
    end
end
IGeP=PreFakt .* IGeP1 .* IGeP2;
IGoP=PreFakt .* IGoP1 .* IGoP2;

HIGP=NormE*IGeP + 1i*IGoP;
res=angle(HIGP)+pi;
figure(7)
h=surf(R,P,res);
set(h, 'edgecolor','none');
d=d*vertcat(zeros(1,size(res,2)),diff(res));  %difference: d/d(phi)
for i=1:size(d,1)  %remove artificial jumps
    for j=1:size(d,2)
        if abs(d(i,j))>0.5
            if i>1
                d(i,j)=d(i-1,j);
            else
                d(i,j)=d(i+1,j);
            end
        end
    end
end

d=d*(size(d,2)/(2*pi));
ddif=d*(size(d,1)/(2*pi));
```

C.1 Ince-Gauss calculations

```matlab
figure(8)
h=surf(R,P,ddif);
set(h, 'edgecolor','none');
intensity=abs(HIGP).^2;
for i=2:(nx-1)
    for j=2:(ny-1)
        a=intensity(i-1,j);
        b=intensity(i+1,j);
        c=intensity(i,j);
        if c>max(a,b) || c<min(a,b)
            intensity(i,j)=(a+b)/2;
        end
    end
end
intensity=intensity/(sum(sum(intensity)));
dlmwrite(['savevalue/ig/int',num2str(nx),',',num2str(ny),',', ...
    num2str(tmpRot),',',num2str(tempP),',',num2str(tmpM),',', ...
    num2str(elips)], intensity, 'precision', 5);
dlmwrite(['savevalue/ig/d',num2str(nx),',',num2str(ny),',', ...
    num2str(tmpRot),',',num2str(tempP),',',num2str(tmpM),',', ...
    num2str(elips)], ddif, 'precision', 5);
if (exist(['savevalue/ig/dxy',num2str(nx),',',num2str(ny),',', ...
    num2str(tmpRot),',',num2str(tempP),',',num2str(tmpM),',', ...
    num2str(elips)], 'file') && exist(['savevalue/ig/intxy',num2str(nx), ...
    num2str(tmpRot),',',num2str(tempP),',',num2str(tmpM),',', ...
    num2str(elips)], 'file'))
    dxy=dlmread(['savevalue/ig/dxy',num2str(nx),',',num2str(ny),',', ...
        num2str(tmpRot),',',num2str(tempP),',',num2str(tmpM),',', ...
        num2str(elips)]);
    intxy=dlmread(['savevalue/ig/intxy',num2str(nx),',',num2str(ny), ...'
        num2str(tmpRot),',',num2str(tempP),',',num2str(tmpM),',', ...
        num2str(elips)]);
else
    dxy=zeros(size(X,1),size(Y,2));
    intxy=zeros(size(X,1),size(Y,2));
    for i=1:size(X,1)
        for j=1:size(Y,2)
            xx=X(i,j);
            yy=Y(j,i);
            rr=sqrt(xx^2+yy^2);
            pp=atan2(yy,xx)+pi;
            [tmp,r_i]=min(abs(R-rr));  % Index for R
            [tmp,p_i]=min(abs(P-pp));  % Index for Phi
            if (R(r_i,j)-rr)>0
                r_i2=r_i;
            end
            if r_i>l
```
C.1 Ince-Gauss calculations

\[
\begin{align*}
    r_{i1} &= r_{i} - 1; \\
    \text{else} & \quad r_{i1} = r_{i}; \\
    \text{end} \\
    r_{i1} &= r_{i}; \\
    \text{if} \ r_{i} < \text{size}(R, 2) & \quad r_{i2} = r_{i} + 1; \\
    \text{else} & \quad r_{i2} = r_{i}; \\
    \text{end} \\
    \text{end} \\
    \text{else} & \quad r_{i1} = r_{i}; \\
    \text{if} \ r_{i} < \text{size}(P, 2) & \quad p_{i2} = p_{i}; \\
    \text{else} & \quad p_{i1} = p_{i}; \\
    \text{end} \\
    \text{if} \ (P(p_{i}) - pp) > 0 & \quad p_{i1} = p_{i}; \\
    \text{else} & \quad p_{i1} = p_{i}; \\
    \text{end} \\
    \text{end} \\
\end{align*}
\]

\[
\begin{align*}
    I_{11} &= (1 - \sqrt{(R(r_{i1}) - rr)^2 / (R(r_{i2}) - R(r_{i1}))^2}) \times ... \\
    & \quad (1 - \sqrt{(P(p_{i1}) - pp)^2 / (P(p_{i2}) - P(p_{i1}))^2}); \\
    I_{12} &= (1 - \sqrt{(R(r_{i1}) - rr)^2 / (R(r_{i2}) - R(r_{i1}))^2}) \times ... \\
    & \quad (1 - \sqrt{(P(p_{i1}) - pp)^2 / (P(p_{i2}) - P(p_{i1}))^2}); \\
    I_{21} &= (1 - \sqrt{(R(r_{i2}) - rr)^2 / (R(r_{i1}) - R(r_{i2}))^2}) \times ... \\
    & \quad (1 - \sqrt{(P(p_{i2}) - pp)^2 / (P(p_{i1}) - P(p_{i2}))^2}); \\
    I_{22} &= (1 - \sqrt{(R(r_{i2}) - rr)^2 / (R(r_{i1}) - R(r_{i2}))^2}) \times ... \\
    & \quad (1 - \sqrt{(P(p_{i2}) - pp)^2 / (P(p_{i1}) - P(p_{i2}))^2}); \\
    dxy(i, j) &= I_{11} \times \text{ddif}(p_{i1}, r_{i1}) + I_{12} \times \text{ddif}(p_{i2}, r_{i1}) + ... \\
    & \quad I_{21} \times \text{ddif}(p_{i1}, r_{i2}) + I_{22} \times \text{ddif}(p_{i2}, r_{i2}); \\
    intxy(i, j) &= I_{11} \times \text{intensity}(p_{i1}, r_{i1}) + I_{12} \times \text{intensity}(p_{i2}, r_{i1}) \ldots \\
    & \quad I_{21} \times \text{intensity}(p_{i1}, r_{i2}) + I_{22} \times \text{intensity}(p_{i2}, r_{i2}); \\
\end{align*}
\]

\[
\begin{align*}
    \text{dlmwrite}([''savevalue/ig/intxy'',' num2str(nx), ',',' num2str(ny), ',' ,', ... \\
    \text{num2str(tmpRot), ',',' num2str(tmpP), ',' ,', ... \\
    \text{num2str(elips)]}, \text{intxy}, 'precision', 5); \\
    \text{dlmwrite}([''savevalue/ig/dxy'',' num2str(nx), ',',' num2str(ny), ',' ,', ... \\
    \text{num2str(tmpRot), ',',' num2str(tmpP), ',' ,', ... \\
    \text{num2str(elips)]}, \text{dxy}, 'precision', 5); \\
\end{align*}
\]

\[
\begin{align*}
    \text{figure}(9)
\end{align*}
\]
C.1 Ince-Gauss calculations

h=surf(R,P,ddif,intensity);
set(h, 'edgecolor','none');
view(310,55)
axis([0,500,0,8,−2000,2000])
saveas(9,strcat('savevalue\rp2\hig',int2str(x00+1000) ',',int2str(y00+1000) ),'.png', 'png')
figure(10)
h=surf(X(:,1),Y(:,1),dxy,intxy);
set(h, 'edgecolor','none');
view(335,70)
axis([-200,200,-200,200,−2000,2000])
saveas(10,strcat('savevalue\xyhig',int2str(100*elips),'.png'), 'png')
d=ddif*2*pi/size(ddif,2);
DL=sum(sum(d.*intensity));
fprintf('DL: %f

',DL);
fwrite(fID, [num2str(x00) ' ' num2str(y00) ' ' num2str(DL) '
']);
figure(1)
end

% fctCartesianToElliptic
% Transforms Cartesian Coordinates to elliptic Coordinates
% The inverse Transformation to:
% x=a*Cosh(u)*Cos(v)
% y=a*Sinh(u)*Sin(v)
% In:
% X ... Matrix with x values in row
% Y ... Matrix with y values in columns
% a ... Constant
% Out:
% u ... x*y matrix of coordinate 1, u is real and [0,inf)
% v ... x*y matrix of coordinate 2, v is real and [0,2Pi)
% Transformation functions with Mathematica command:
% Solve[{x == a*Cosh[xi] *Cos[eta], y == a*Sinh[xi] *Sin[eta]}, {xi, eta}]

function [u, v] = fctCartesianToElliptic(X,Y,a)
    u=zeros(size(Y,1),size(X,2));
    v=zeros(size(Y,1),size(X,2));
    for ix=1:size(X,2)
        x=X(:,ix);
        if (x==0)
            x=1e−5; % no 'div by zero' error
        end
        for iy=1:size(Y,1)
            % Calculation...
        end
    end
y = Y(iy, 1);
sol = zeros(16, 2);
wl = sqrt(4*aˆ2*yˆ2+(aˆ2-xˆ2-yˆ2)ˆ2); % Inner Root
w2 = sqrt(1+(xˆ2/aˆ2)+(yˆ2/aˆ2)-(w1/aˆ2)); % Outer Root
t1 = (x+w2)/(2*sqrt(2));
t2 = (x^-2*(2*a))/2*sqrt(2)*a);
t3 = (y+w2)/(2*sqrt(2)*a);
t4 = (w1+w2)/(2*sqrt(2)*a);
tall1 = t1 + t2 + t3 + t4;
s1 = w2/sqrt(2);
sol(1,1) = acos(-tall1/x); sol(1,2) = acosh(-s1);
sol(2,1) = acos(-tall1/x); sol(2,2) = acosh(-s1);
sol(3,1) = acos(-tall1/x); sol(3,2) = acosh(-s1);
sol(4,1) = acos(-tall1/x); sol(4,2) = acosh(-s1);
sol(5,1) = acos(+tall1/x); sol(5,2) = acosh(+s1);
sol(6,1) = acos(+tall1/x); sol(6,2) = acosh(+s1);
sol(7,1) = acos(+tall1/x); sol(7,2) = acosh(+s1);
sol(8,1) = acos(+tall1/x); sol(8,2) = acosh(+s1);
w3 = sqrt((1/2)+(xˆ2/(2*aˆ2))+(yˆ2/(2*aˆ2))+(w1/(2*aˆ2)));
t5 = (1/2)*a*w3;
t6 = (xˆ2/(2*a))*w3;
t7 = (yˆ2/(2*a))*w3;
t8 = (w1+w3)/(2*a);
tall2 = t5 + t6 + t7 - t8;
s2 = w3;
sol(9,1) = acos(+tall2/x); sol(9,2) = acosh(+s2);
sol(10,1) = acos(+tall2/x); sol(10,2) = acosh(+s2);
sol(11,1) = acos(+tall2/x); sol(11,2) = acosh(+s2);
sol(12,1) = acos(+tall2/x); sol(12,2) = acosh(+s2);
sol(13,1) = acos(-tall2/x); sol(13,2) = acosh(-s2);
sol(14,1) = acos(-tall2/x); sol(14,2) = acosh(-s2);
sol(15,1) = acos(-tall2/x); sol(15,2) = acosh(-s2);
sol(16,1) = acos(-tall2/x); sol(16,2) = acosh(-s2);
for j = 1:16
    if (abs(imag(sol(j,1)))<1e-5 && abs(imag(sol(j,2)))<1e-5) ... % Just real solutions
        if (sol(j,2)>0 && ... 
            sign(sinh(sol(j,2))*sin(sol(j,1))) == sign(y)) % ... 
            u = [0, inf], right sign for y (which is not determined ... 
                by the algorithm 
            u(iy, ix) = sol(j,2); 
            v(iy, ix) = sol(j,1) + (1 - sign(sol(j,1)))*pi; ... 
                % Matlab solution: [-pi, pi] ... 
            -> [0, 2pi]
        end
    end
% fctSolveRecurrenceRelation
% Solves the recurrence relation for the coefficients of the ansatz to the
% Ince differential equation
%
% Input:
% eo ... 'e' or 'o', p=2n (even) or p=2n+1 (odd) respecitively
% n ... Integer number that defines the highest term in the ansatz for
% the Ince equation.
% m ... Integer number that defines one specific eta in the set of
% solutions to the eta-coefficient matrix
% xi ... defines the ellipticity of the equation.
% xi=0: Laguerre-Gaussian beam
% xi->inf: Hermite-Gaussian beam
%
% Output:
% Cn ... Array with coefficents (C0, C1, C2, ...)
% For even: A0=C0
% For odd: B0=C0
%
% All information from "Periodic Differential Equations" by F.M. Arscott

function Cn = fctSolveRecurrenceRelation(eo, p, in_m, xi)

n=(p-mod(p,2))/2; % n according to Arscott
m=(in_m-mod(in_m,2))/2; % m according to Arscott
if (mod(p,2)==0)
 mtx=zeros(n+1,n+1);

 % 1st row:
 mtx(1,1)=0 % beta_0=0
 mtx(1,2)=(n+1)*xi;

 % 2nd row:
 mtx(2,1)=2*n*xi;
 mtx(2,2)=4;
 if (n>1)
 mtx(2,3)=(n+r)*xi;
 end

 % (n+1)th row:
 mtx(n+1,n)=xi;
 mtx(n+1,n+1)=4*n^2;
end
end

if (eo=='o')
    mtx(1,:)=[];
    mtx(:,1)=[];
end
[v,d]=eig(mtx);
[trash,k]=sort(real(diag(d))); % Sort them
mtx=v(:,k);

if (eo=='o')
    Cn=[0;v(:,k(m))];
else
    Cn=v(:,k(m+1));
end
else
    mtx=zeros(n+1,n+1);
    % 1st row:
    mtx(1,1)=xi*(n+1)+1; % beta_0=0
    mtx(1,2)=(n+2)*xi;
    % 2nd row:
    mtx(2,1)=n*xi;
    mtx(2,2)=9;
    if (n>1)
        mtx(2,3)=(n+3)*xi;
    end
    for c=3:n
        r=c-2; % Matlab's matrixes start at (1,1), so ...
        % there is a difference between r from formular and c of counter
        mtx(c,c-1)=(n-r)*xi; % alpha_r
        mtx(c,c)=(2*r+3)^2; % beta_r
        mtx(c,c+1)=(n+r+3)*xi; % gamma_r
    end

    % (n+1)th row:
    if (n>1)
        mtx(n+1,n)=xi;
    end
    mtx(n+1,n+1)=(2*n+1)^2;
end
if (eo=='o')
    mtx(1,1)=-xi*(n+1)+1;
end
[v,d]=eig(mtx);
[trash,k]=sort(real(diag(d))); % Sort them
mtx=v(:,k);
Cn=v(:,k(m+1));

% fctIncePolyVal
% Evaluates the Ince polynom. Equivalent to the command 'polyval'.

% Input:
% eo ... 'e' or 'o', p=2n (even) or p=2n+1 (odd) respectively
% p ... Integer number that defines the highest term in the ansatz for
% the Ince equation.
% in_m ... Integer number that defines one specific eta in the set of
% solutions to the eta-coefficient matrix
% xi ... defines the ellipticity of the equation.
% xi=0: Laguerre-Gaussian beam
% xi -> inf: Hermite-Gaussian beam
% X ... 2x2 Array of x-values to be evaluated

% Output:
% InceValue ... Array of Ince(x)

% All information from "Periodic Differential Equations" by F.M. Arscott

function rv = fctIncePolyVal(eo, p, in_m, xi, X, coeff)

n = (p - mod(p, 2))/2; % n according to Arscott
m = (in_m - mod(in_m, 2))/2; % m according to Arscott

% Generate the right base first
if eo == 'e'
    if mod(p, 2) == 0
        % Cos(2r*z)
        for c = 1:n+1
            r = c-1; % r starts at zero, c starts at 1
            base(c,:) = cos(2*r*X);
        end
    else
        % Cos((2r+1)*z)
        for c = 1:n+1
            r = c-1; % r starts at zero, c starts at 1
            base(c,:) = cos((2*r+1)*X);
        end
    end
elseif eo == 'o'
    if mod(p, 2) == 0
        % Sin(2r*z)
        for c = 1:n+1
            r = c-1; % r starts at zero, c starts at 1
            base(c,:) = sin(2*r*X);
        end
    else
        % Sin((2r+1)*z)
        for c = 1:n+1
            r = c-1; % r starts at zero, c starts at 1
            base(c,:) = sin((2*r+1)*X);
        end
    end
end

%fprintf('

');
for xx=1:size(X,2) % not native Matlab style, but didn't...
    for yy=1:size(X,1)
        InceValue(yy,xx)=sum(coeff.*base(:,yy,xx));
    end
end
rv=InceValue;

function [un, vn] = fctPolarToElliptic(R,P,x0,y0,a)
    un=zeros(size(R,2),size(P,2));
    vn=zeros(size(R,2),size(P,2));
    for ix=1:size(R,2)
        for iy=1:size(P,2)
            x=R(ix)*cos(P(iy))-x0;
            y=R(ix)*sin(P(iy))-y0;
            sol=zeros(16,2);
            w1=sqrt(4*a^2*y^2+(a^2-x^2-y^2)^2); % inner Root
            w2=sqrt(1+(x^2/a^2)+(y^2/a^2)-(w1/a^2)); % outer Root
            t1=(a*w2)/(2*sqrt(2));
            t2=(x^2*w2)/(2*sqrt(2)*a);
            t3=(y^2*w2)/(2*sqrt(2)*a);
            t4=(w1*w2)/(2*sqrt(2)*a);
            tall1=t1+t2+t3+t4;
            s1=w2/sqrt(2);
C.2 Maverick patterns

\[ \text{sol}(1,1) = -\cos\left(\frac{-\text{tall1}}{x}\right); \quad \text{sol}(1,2) = -\cosh\left(-\text{s1}\right); \]

\[ \text{sol}(2,1) = -\cos\left(\frac{-\text{tall1}}{x}\right); \quad \text{sol}(2,2) = +\cosh\left(+\text{s1}\right); \]

\[ \text{sol}(3,1) = +\cos\left(\frac{-\text{tall1}}{x}\right); \quad \text{sol}(3,2) = -\cosh\left(-\text{s1}\right); \]

\[ \text{sol}(4,1) = +\cos\left(\frac{-\text{tall1}}{x}\right); \quad \text{sol}(4,2) = +\cosh\left(+\text{s1}\right); \]

\[ \text{sol}(5,1) = -\cos\left(\frac{+\text{tall1}}{x}\right); \quad \text{sol}(5,2) = -\cosh\left(+\text{s1}\right); \]

\[ \text{sol}(6,1) = -\cos\left(\frac{+\text{tall1}}{x}\right); \quad \text{sol}(6,2) = +\cosh\left(+\text{s1}\right); \]

\[ \text{sol}(7,1) = +\cos\left(\frac{+\text{tall1}}{x}\right); \quad \text{sol}(7,2) = +\cosh\left(+\text{s1}\right); \]

\[ \text{sol}(8,1) = +\cos\left(\frac{+\text{tall1}}{x}\right); \quad \text{sol}(8,2) = +\cosh\left(+\text{s1}\right); \]

\[ \text{sol}(9,1) = -\cos\left(\frac{+\text{tall2}}{x}\right); \quad \text{sol}(9,2) = -\cosh\left(+\text{s2}\right); \]

\[ \text{sol}(10,1) = -\cos\left(\frac{+\text{tall2}}{x}\right); \quad \text{sol}(10,2) = +\cosh\left(+\text{s2}\right); \]

\[ \text{sol}(11,1) = +\cos\left(\frac{+\text{tall2}}{x}\right); \quad \text{sol}(11,2) = -\cosh\left(+\text{s2}\right); \]

\[ \text{sol}(12,1) = +\cos\left(\frac{+\text{tall2}}{x}\right); \quad \text{sol}(12,2) = +\cosh\left(+\text{s2}\right); \]

\[ \text{sol}(13,1) = -\cos\left(\frac{-\text{tall2}}{x}\right); \quad \text{sol}(13,2) = -\cosh\left(-\text{s2}\right); \]

\[ \text{sol}(14,1) = -\cos\left(\frac{-\text{tall2}}{x}\right); \quad \text{sol}(14,2) = +\cosh\left(-\text{s2}\right); \]

\[ \text{sol}(15,1) = +\cos\left(\frac{-\text{tall2}}{x}\right); \quad \text{sol}(15,2) = -\cosh\left(-\text{s2}\right); \]

\[ \text{sol}(16,1) = +\cos\left(\frac{-\text{tall2}}{x}\right); \quad \text{sol}(16,2) = +\cosh\left(-\text{s2}\right); \]

\[ \text{for} \ j = 1:16 \]
\[ \quad \text{if (abs(\text{imag(sol(j,1))) < 1e-5 && abs(\text{imag(sol(j,2))) < 1e-5}) ...} \]
\[ \qquad \% \text{Just real solutions} \]
\[ \qquad \text{if (sol(j,2) > 0 && ...} \]
\[ \qquad \quad \text{sign(\text{sinh(sol(j,2))) } \ast \text{sin(sol(j,1))) } = \text{sign(y)} \% \text{...} \]
\[ \qquad \quad \text{u=0,inf}, \text{right sign for y (which is not determined ...} \]
\[ \qquad \quad \text{by the algorithm} \]
\[ \qquad \quad \text{vn(iy, ix)=sol(j,2);} \]
\[ \qquad \quad \text{vn(iy, ix)=sol(j,1)+(1-sign(sol(j,1)))*pi; ...} \]
\[ \qquad \quad \% \text{Matlab solution: [-pi,pi] ...} \]
\[ \qquad \quad \text{-> [0, 2pi]} \]
\[ \quad \text{end} \]
\[ \text{end} \]
\[ \text{end} \]

C.2. Maverick patterns
C.2 Maverick patterns

```matlab
4 tmpL=0;
5 tmpP=0;
6 if(size(tmpFind,2)==0)
7     disp('Wrong format of MAV input. Should have the form: ...
8         mavP,l1,l2,...,\Delta P − l defines the 2Pi multiplicity and \Delta P gives ...
9         an additive phase');
10 else
11     tmpP=real(str2double(C(4:tmpFind(1)-1)));
12     if (size(tmpFind,2) \neq (tmpP+1)*2)
13         disp('Wrong format of MAV input. Should have the form: ...
14             mavP,l1,l2,...,\Delta P − l defines the 2Pi multiplicity and \Delta P ... gives an additive phase');
15     else
16         L_gen=[];
17         \Delta=[];
18         count=1;
19         while count<size(tmpFind,2)-2
20             L_gen=[L_gen ... 
21                 real(str2double(C(tmpFind(count):tmpFind(count+1)-1)))];
22             \Delta=[\Delta real(str2double(C(tmpFind(count+1):tmpFind(count+2)-1)))];
23             count=count+2;
24         end
25         L_gen=[L_gen real(str2double(C(tmpFind(count):tmpFind(count+1)-1)))];
26         \Delta=[\Delta real(str2double(C(tmpFind(count+1):end)))];
27         L_gen
28         \Delta
29         \%holo(gcf).data=angle(polyval(LaguerreGen(tmpP, abs(tmpL)), ...
30            2*(Rad.*Rad)/(w0)ˆ2));
31         rootsL=sort([(sqrt(roots(LaguerreGen(tmpP, abs(L_gen(1))))/2)*w0)' ...
32             max(max(Rad))+1]);
33         holo(gcf).data=angle((Rad < rootsL(1)) .* exp(1i * ...
34             (L_gen(1)*Angle + \Delta(1) * (pi/180))));
35         count=2;
36         while count<tmpP+1
37             holo(gcf).data=holo(gcf).data+(Rad < rootsL(count-1)) .* ... 
38                 (Rad < rootsL(count)) .* (angle(exp(1i * (L_gen(count) * ...
39                     Angle+\Delta(count) * (pi/180)))));
40             count=count+1;
41         end
42     end
43     \% Maverick mode superpositions
44 else
45     disp('Wrong format of MAVSUP input. Should have the form: ...
46         mavsupPhase,P,l1,l2,...,\Delta P − l defines the 2Pi multiplicity and ... 
47         \Delta P gives an additive phase');
48     end
49 end
```
else
    phase=real(str2double(C(7:tmpFind(1)-1))+180;
    tmpP=real(str2double(C(tmpFind(1):tmpFind(2)-1)));
    if (size(tmpFind,2)-tmpP+1) == 1
        disp('Wrong format of MAVSUP input. Should have the form: ...
            mavsupPhase,P,l1,a1,l2,...,aP − l defines the 2Pi multiplicity ...
            and aP gives an additive phase');
    else
        L_gen=[];
        count=2;
        while count<size(tmpFind,2)-2
            L_gen=[L_gen ...
                real(str2double(C(tmpFind(count):tmpFind(count+1)-1)))];
            A=[A real(str2double(C(tmpFind(count+1):tmpFind(count+2)-1)))];
            count=count+2;
        end
        L_gen=[L_gen real(str2double(C(tmpFind(count):end)))];
    end
    %holo(gcf).data=angle(polyval(LaguerreGen(tmpP, abs(tmpL)), ...
        2*(Rad.*Rad)/(w0)ˆ2));
    rootsL=sort([sqrt(roots(LaguerreGen(tmpP, abs(L_gen))))/2]*w0)'; ...
    max(max(Rad)+1);
    tmpdataP=(Rad<rootsL(1)).*exp(+1i*(L_gen(1)*Angle-Δ(1)*(pi/180)));
    tmpdataM=(Rad<rootsL(1)).*exp(-1i*(L_gen(1)*Angle-Δ(1)*(pi/180)));
    count=2;
    while count<=tmpP+1
        tmpdataP=tmpdataP+(Rad>=rootsL(count-1)) .* (Rad<rootsL(count)) ... 
            .* (exp(+1i * (L_gen(count)*Angle*Δ(count) * (pi/180))));
        tmpdataM=tmpdataM+(Rad>=rootsL(count-1)) .* (Rad<rootsL(count)) ... 
            .* (exp(-1i * (L_gen(count)*Angle+Δ(count) * (pi/180))));
        count=count+1;
    end
    holo.data=angle(exp(+1i*phase*(pi/180)) * tmpdataP+exp(-1i * phase ... 
        * (pi/180)) * tmpdataM);
end
end

C.3. Vortex cut-out
C.4 Random pattern

8 \text{AntiIntensity} = (\text{abs}(\text{real}(\text{HelicialIntensity})) \leq 0.05 \pi) \cdot \ldots \\
\text{abs}(\text{imag}(\text{HelicialIntensity})) \leq 0.05 \pi); \\
B = \text{AntiIntensity}(300,:); \\
ivortex = []; \\
\text{while } c == 0 \\
\hspace{1em} iold = i; \\
\hspace{1em} B(i) = 2; \text{ \% make unzero} \\
\hspace{1em} [c, i] = \text{min}(B); \\
\hspace{1em} \text{if } |i - iold + 1| \land (c == 0) \\
\hspace{2em} \text{ivortex} = [\text{ivortex} \text{ \% round}((iold + i)/2)]; \\
\hspace{1em} \end \\
\end \\
\text{end} \\
\text{ivortex} \\

[Xt, Yt] = \text{meshgrid}(y + (396 - \text{ivortex}(vn)), x); \\
\text{Radt} = \sqrt{Xt.^2 + Yt.^2}; \\
\text{C} = ((\text{Radt}) \leq \text{rsize}); \\
\text{holo(gcf).data} = \text{holo(gcf).data} \cdot C; \\
\text{holo(gcf).grat} = \text{holo(gcf).grat} \cdot C; \\
\text{if (gcf == 2)} \\
\hspace{1em} \text{holo(gcf).grat} = \text{grating}(0, 2\pi/10, X, Y); \\
\hspace{1em} \text{holo(gcf).grat} = \text{holo(gcf).grat} \cdot \text{fliplr}(C); \\
\hspace{1em} \text{holo(gcf).data} = \text{fliplr(holo(gcf).data);} \\
\end \\
\end \\
\text{ivortex} \\

C.4. Random pattern

1 \text{elseif strncmp(C,'RandomPattern',12) == 1 && isempty(C(3:end)) == 0;} \\
2 \text{count} = 0; \\
3 \text{holosize} = \text{size(holo(1).data);} \\
4 \text{tmp} = \text{zeros(holosize(1), holosize(2));} \\
5 \text{while } \text{count} < 1000 \\
6 \hspace{1em} y = \text{randi(holosize(1) - 20);} \\
7 \hspace{1em} ysize = \text{randi([20, min(holosize(1) - y, 115)]);} \\
8 \hspace{1em} x = \text{randi(holosize(2) - 20);} \\
9 \hspace{1em} xsize = \text{randi([20, min(holosize(2) - x, 115)]);} \\
10 \hspace{1em} \text{tmp}(y:y+ysize, x:x+xsize) = \text{rand}() \cdot 2\pi; \\
11 \hspace{1em} \text{count} = \text{count} + 1; \\
\end \\
\text{rndpattern} = \text{tmp}; \\
\text{invarandompattern} = (2\pi - \text{tmp}); \\
\text{holo(gcf).data} = \text{rndpattern}; \\
\text{if (gcf == 2)} \\
\hspace{1em} \text{holo(gcf).data} = \text{fliplr(holo(gcf).data);} \\
\end \\
\end \\
\text{holo(1).data} = \text{rndpattern}; \\
\text{holo(2).data} = \text{fliplr(invarandompattern);}
else if strncmp(C, 'DisplayRandomReal', 17)==1
   if exist('rndpattern')==0
      count=0;
      holosize=size(holo(1).data);
      tmp=zeros(holosize(1),holosize(2));
      while count<1000
         y=randi(holosize(1)-20);
         ysize=randi([20,min(holosize(1)-y,115)]);
         x=randi(holosize(2)-20);
         xsize=randi([20,min(holosize(2)-x,115)]);
         tmp(y:y+ysize,x:x+xsize)=rand()*2*pi;
         count=count+1;
      end
      rndpattern=tmp;
      invrandompattern=(2*pi-tmp);
   end
   holo(gcf).data=rndpattern;
   if (gcf==2)
      holo(gcf).data=fliplr(holo(gcf).data);
   end
end
else if strncmp(C, 'DisplayRandomInvert', 19)==1
   if exist('invrandompattern')==1
      holo(gcf).data=invrandompattern;
   else
      disp('Does not exist, call rndpattern first!')
   end
   if (gcf==2)
      holo(gcf).data=fliplr(holo(gcf).data);
   end
Acknowledgements

Zuerst möchte ich Prof. Anton Zeilinger danken für die tolle Möglichkeit in seiner Gruppe an diesem interessanten Projekt zu arbeiten. Die Diskussionen über verschiedenste physikalische Fragestellungen haben mir öfters eine ganz neue Sichtweise gegeben.

Meinem Betreuer Robert Fickler danke ich vielmals für sein immenses Engagement und die unermüdliche Motivation mir mit jedem auftretenden Problem zu helfen. Das hat nicht nur dazu geführt, dass die Experimente erfolgreich durchgeführt werden konnten, sondern vor allem auch dass die Arbeit viel Spaß gemacht hat.


Radek Lapkiewicz gebührt mein Dank unter anderem für die vielen Erklärungen zu grundlegenden quantenmechanischen Themen und die vielen teils sehr unkonventionellen und kreativen Lösungsvorschläge und Ideen, und vor allem auch für das kontinuierliche Verbreiten von guter Laune.

Thanks to Bill Plick for the numerous helpful and interesting discussions about various topics in connection with this work, and several clever and useful theoretical suggestions and advices.

Christoph Schöff und Robert Polster möchte ich für die interessanten, hilfreichen und manchmal ausgefallenen Diskussionen danken. Das hat immer gute Stimmung gemacht und so allgemein die Motivation gehoben.

Marcus Huber möchte ich danken für die sehr kompetente und unermüdliche Hilfe beim Thema Entanglement Detection. Durch die Korrespondenz habe ich den mathematischen Formalismus kennen gelernt, was sehr nützlich ist.

Last but not least geht mein grösster Dank an meine Familie für die kontinuierliche bedingungslose finanzielle und moralische Unterstützung. Vielen Dank an meine Eltern Magrit und Herbert Krenn, meine Oma Theresia Zechner und meine Geschwister Oliver und Aline Krenn!
Bibliography


