Quantum Physics in Lean

Mario Krenn*

Max Planck Institute for the Science of Light, Erlangen, Germany.

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1 Introduction

The goal of this project is to introduce in Lean mathematical concepts that play important roles in quantum optics [2, 3].

2 Graphs with only disjoint perfect matchings (Bogdanov's Lemma)

Let's consider (not necessary simple) graphs with only disjoint perfect matchings (i.e. every edge only appears in at most one of the perfect matchings).
Then, if such a graph has 4 vertices, it can have maximally three perfect matchings (K_4 is an example). If it has more than 4 vertices, it can have maximally two perfect matchings (C_{2n} is an example). The proof by Ilya Bogdanov is here [1].

3 Generalization to weighted colored graphs: Into quantum physics

Next, we generalize the statement of Bogdanov's lemma to weighted and edge-colored graph, which are used as a representation in quantum optics. More details in a MO I and MO II, Dustin Mixon's blog, youtube video.

Definition 1: Edge-colored weighted graphs
Let's consider a (not necessary simple) graph G(V,E). The vertices are labeled from $v_1, \dots v_{|V|}$. Every edge $e \in E$ has associated with it a color from a set of d different colors. Furthermore, every edge carries a complex weight $w_e \in \mathbb{C}$.[1]

Definition 2: Perfect matching (PM)
A perfect matching is a subset of E which contains every vertex $v \in V$ exactly once.

*mario.krenn@mpl.mpg.de, Artificial Scientist Lab.

[1]A generalization are bi-chromatic graphs: Every edge between vertices v_i and v_j has two (not necessarily different) colors (c_i and c_j), with c_i incident at vertex v_i and c_j incident at vertex v_j. This property could be left out for the moment.

Definition 3: Weight of Perfect matching (PM)
The weight w_{PM} of a PM is defined as the product of the weights of its edges w_e, i.e.

$$w_{\mathrm{PM}} = \prod_{e \in PM} w_e.$$

Definition 4: Induced vertex coloring (ivc)
Let G be an edge-colored weighted graph and PM denote a perfect matching in G. We define a vertex coloring c that is induced by the color of the incident (colored) edges of the PM. We write ivc(PM)=c. We call a vertex coloring, in which every vertex has the same color, monochromatic. Otherwise, we call it polychromatic.

Definition 5: Weight of vertex coloring
Let $\mathcal{M}$(c) be the set of all PMs p of a graph G with ivc(p)=c. The weight of a vertex coloring c is the sum of weights of all PMs in $\mathcal{M}(c)$, i.e.,

$$w(\mathrm{c}) = \sum_{\forall p \in \mathcal{M}(c)} \prod_{e \in p} w_e. \tag{1}$$

If $\mathcal{M}$(c)=$\emptyset$, then $w(\mathrm{c}) = 0$.

Definition 6: Monochromatic Graph
Let G be an edge-colored graph with complex weights, M_{col} be the set of all monochromatic vertex colorings of G, and P_{col} be the set of all polychromatic vertex colorings of G.
We call G monochromatic, if $\forall c \in M_{col} : w(c) = 1$ and $\forall c \in P_{col} : w(c) = 0$.

Conjecture: Bound of non-zero colorings in monochromatic Graph Let $G(V, E)$ be a monochromatic graph with finite complex weights, and col(G) is the number of different vertex colorings c of G with non-zero weight $w(c)$.

Then, col(G) is bounded by

$$col(G) \leq \begin{cases} 3, & \text{if } |V| = 4 \\ 2, & \text{if } |V| > 4. \end{cases} \tag{2}$$

Proof for w_e in $\mathbb{R}_+$
If all weights w_e are real and positive, the bound of col(G) in (2) holds due to Bogdanov's lemma. Every disjoint perfect matching can have a monochromatic coloring, Bogdanov's lemma gives the bound on the number of disjoint perfect matchings, every other coloring will be a combination of different colors. The coloring weights $w(c)$ of polychromatic colorings cannot cancel as the edge weights $w_e \in \mathbb{R}_+$.

References

[1] Ilya Bogdanov (https://mathoverflow.net/users/17581/ilya bogdanov). Graphs with only disjoint perfect matchings. https://mathoverflow.net/q/267013 (version: 2017-04-12).

[2] Mario Krenn, Xuemei Gu, and Anton Zeilinger. Quantum experiments and graphs: Multiparty states as coherent superpositions of perfect matchings. *Physical review letters*, 119(24):240403, 2017.

[3] Mario Krenn, Jakob S Kottmann, Nora Tischler, and Alán Aspuru-Guzik. Conceptual understanding through efficient automated design of quantum optical experiments. *Physical Review X*, 11(3):031044, 2021.