Perfect Matchings and Quantum Physics

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GHZ States

Greenberger, Horne and Zeilinger(GHZ) studied what might happen if more than two particles are entangled [2]. Such states in which 3 parties are entangled

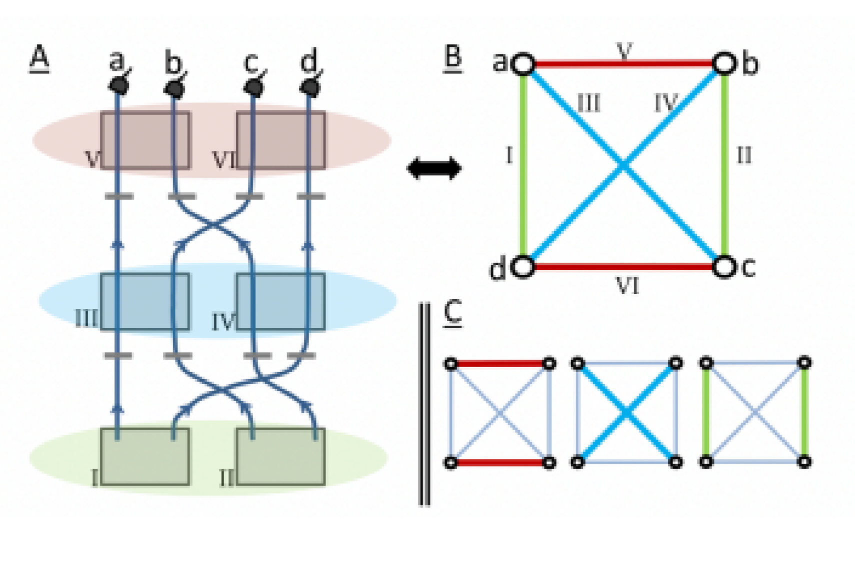
$$|GHZ_{3,2}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

were observed, rejecting local-realistic theories [1, 10].

GHZ states have cryptographic applications in the Quantum Byzantine agreement and are also used in the communication protocols in Distributed quantum computing [6]. Increasing the number of particles involved and the dimension of the GHZ state is essential both for foundational studies and practical applications. A huge effort is being made by several experimental groups around the world to push the size of GHZ states. Photonic technology is one of the key technologies used to achieve this goal [5].

The Graph Theory Connection

In 2017, Krenn et al. [9, 4, 3] discovered a previously hidden bridge between such quantum optical experiments to create high dimensional GHZ states and graph theory. The question "Can high-dimensional GHZ states be created through quantum optical experiments with probabilistic photon sources and linear optics?" reduces to asking if there are edge-coloured edge-weighted graphs with *certain* properties.



Correspondence between experiment and graph

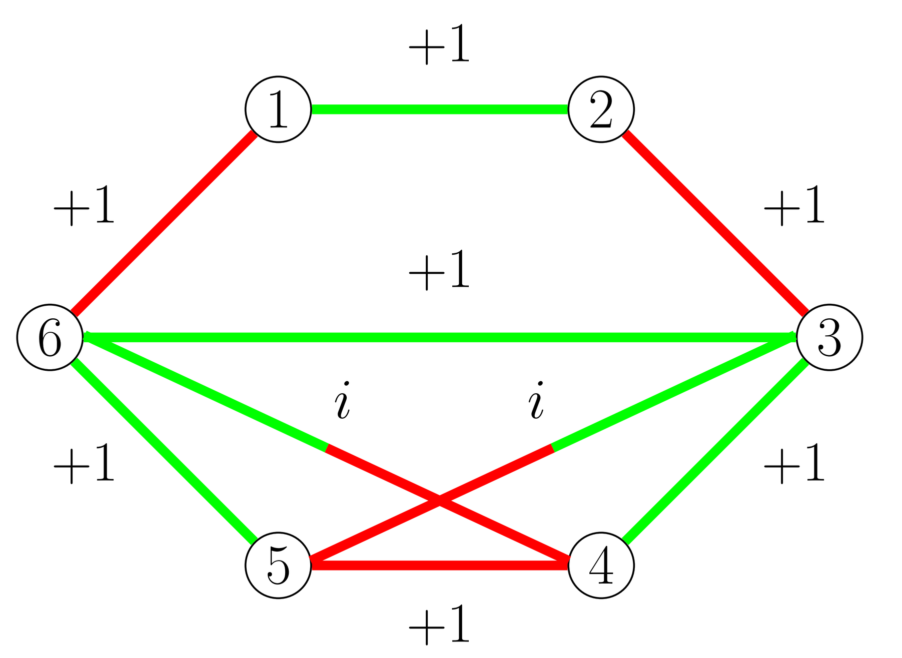
Perfectly Monochromatic Graphs

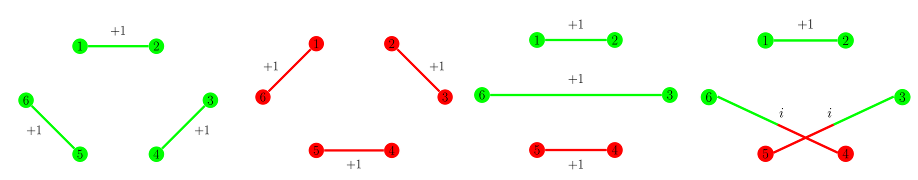
Weight of a Perfect Matching: The weight of a perfect matching is defined as the product of the weights of the edges.

Weight of a Vertex Colouring: The weight of a vertex colouring is defined as the sum of the weights of the perfect matchings which induce the vertex colouring.

Perfectly Monochromatic Graph: We say that an edge-coloured edge-weighted graph is perfectly monochromatic if the weight of all monochromatic vertex colourings is 1 and the weight of all non-monochromatic vertex colourings is 0.

Matching Index: For a simple graph G, the matching index $\mu(G)$ is the maximum number of different colours for which there are monochromatic vertex colourings with weight 1 over all perfectly monochromatic multigraphs with G as the underlying simple graph.



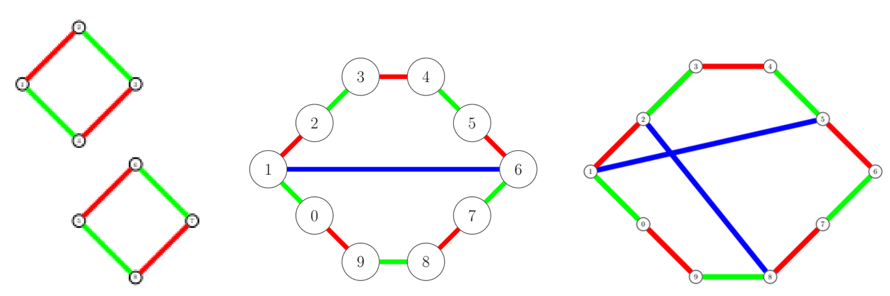


Perfectly Monochromatic Graph

Krenn's Conjecture: $\mu(K_4) = 3$ and for a graph G which is non-isomorphic to K_4, $\mu(G) \leq 2$.

Destructive Interference

When there is no destructive interference, the problem reduces to finding edge-coloured graphs where all perfect matchings are monochromatic. Due to a result by Bogdanov [9], we know that except for K_4, all graphs can have unweighted matching $\bar{\mu}$ index at most 2. The unweighted matching index of K_4 is 3.



Bogdanov's proof

Based on this we can classify graphs into 4 types depending upon whether their unweighted matching index is 0, 1, 2 or 3. It is interesting to ask if destructive interference helps in each of these cases.

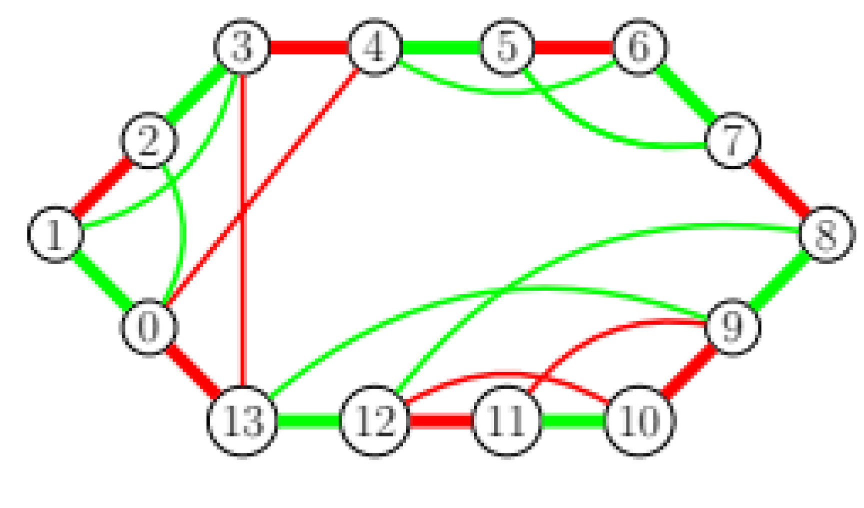
If $\bar{\mu}(G) = 0$, $\mu(G) = 0$

If $\bar{\mu}(G) = 1$, $\mu(G)$ need not be 1. It is unknown whether it can go above 2

If $\bar{\mu}(G) = 3$, $G = K_4$. $\mu(K_4) = 3$. This was proved by Kevin [7] using Groebner bases. This was a computer proof.

Our Results

We gave a complete structural characterisation of graphs corresponding to experiments which create GHZ states of dimension 2 without destructive interference



Structure of graphs with $\bar{\mu} = 2$

We showed that even with destructive interference these graphs can create GHZ states of dimension at most 2

If $\bar{\mu}(G) = 2$, $\mu(G) = 2$

In view of our results, Krenn's conjecture remains open only for graphs with $\bar{\mu} = 1$

References

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